

NUMERICAL ANALYSIS FOR GENERALIZED FORCHHEIMER FLOWS OF SLIGHTLY COMPRESSIBLE FLUIDS IN POROUS MEDIA

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Abstract. In this paper, we will consider the generalized Forchheimer flows for slightly compressible fluids. Using Muskat's and Ward's general form of Forchheimer equations, we describe the fluid dynamics by a nonlinear degenerate parabolic equation for density. The long time numerical approximation of the nonlinear degenerate parabolic equation with time dependent boundary conditions is studied. The stability for all positive time is established in both a continuous time scheme and a discrete backward Euler scheme. A Gronwall's inequality-type is used to study the asymptotic behavior of the solution. Error estimates for the solution are derived for both continuous and discrete time procedures. Numerical experiments confirm the theoretical analysis regarding convergence rates.

Key words. Porous media, immiscible flow, error analysis, Galerkin FEM, nonlinear degenerate parabolic equations, generalized Forchheimer equations, numerical analysis.

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1. Introduction . The most common equation to describe fluid flows in porous media is the Darcy law

$$-\nabla p = \frac{\mu}{\kappa} v, \quad (1.1)$$

where p , v , μ , κ are, respectively (resp.), the pressure, velocity, absolute viscosity and permeability.

When the Reynolds number is large, Darcy's law becomes invalid, see [30, 3]. A nonlinear relationship between the velocity and gradient of pressure is introduced by adding the higher order terms of velocity to Darcy's law. Forchheimer established this in [14] the following three nonlinear empirical models:

$$-\nabla p = av + b|v|v, \quad -\nabla p = av + b|v|v + c|v|^2v, \quad -\nabla p = av + d|v|^{m-1}v, \quad m \in (1, 2). \quad (1.2)$$

Above, the positive constants a, b, c, d are obtained from experiments.

The generalized Forchheimer equation of (1.1) and (1.2) were proposed in [2, 15, 16] of the form

$$-\nabla p = \sum_{i=0}^N a_i |v|^{\alpha_i} v. \quad (1.3)$$

These equations are analyzed numerically in [9, 31, 26], theoretically in [2, 16, 17, 22, 18, 19] for single phase flows, and also in [20, 21] for two-phase flows.

In order to take into account the presence of density in the generalized Forchheimer equation, we modify (1.3) using the dimensional analysis by Muskat [30] and Ward [35]. They proposed the following equation for both laminar and turbulent flows in porous media:

$$-\nabla p = F(v^\alpha \kappa^{\frac{\alpha-3}{2}} \rho^{\alpha-1} \mu^{2-\alpha}), \quad \text{where } F \text{ is a function of one variable.} \quad (1.4)$$

In particular, when $\alpha = 1, 2$, Ward [35] established from experimental data that

$$-\nabla p = \frac{\mu}{\kappa} v + c_F \frac{\rho}{\sqrt{\kappa}} |v|v, \quad \text{where } c_F > 0. \quad (1.5)$$

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Combining (1.3) with the suggestive form (1.4) for the dependence on ρ and v , we propose the following equation

$$-\nabla p = \sum_{i=0}^N a_i \rho^{\alpha_i} |v|^{\alpha_i} v, \quad (1.6)$$

where $N \geq 1$, $\alpha_0 = 0 < \alpha_1 < \dots < \alpha_N$ are real numbers, the coefficients a_0, \dots, a_N are positive. Here, the viscosity and permeability are considered constant, and we do not specify the dependence of a_i 's on them.

Multiplying both sides of the previous equation to ρ , we obtain

$$g(|\rho v|) \rho v = -\rho \nabla p, \quad (1.7)$$

where the function g is a generalized polynomial with non-negative coefficients. More precisely, the function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is of the form

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \dots + a_N s^{\alpha_N}, \quad s \geq 0, \quad (1.8)$$

where $N \geq 1$, $\alpha_0 = 0 < \alpha_1 < \dots < \alpha_N$ are fixed real numbers, the coefficients a_0, \dots, a_N are non-negative numbers with $a_0 > 0$ and $a_N > 0$.

For slightly compressible fluids, the state equation is

$$\frac{d\rho}{dp} = \frac{\rho}{\kappa}, \quad (1.9)$$

which yields

$$\rho \nabla p = \kappa \nabla \rho. \quad (1.10)$$

It follows from (1.8) and (1.10) that

$$g(|\rho v|) \rho v = -\kappa \nabla \rho. \quad (1.11)$$

Solving for ρv from (1.11) gives

$$\rho v = -\kappa K(|\kappa \nabla \rho|) \nabla \rho, \quad (1.12)$$

where the function $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined for $\xi \geq 0$ by

$$K(\xi) = \frac{1}{g(s(\xi))}, \quad \text{with } s = s(\xi) \text{ being the unique non-negative solution of } sg(s) = \xi. \quad (1.13)$$

The continuity equation is

$$\phi \rho_t + \operatorname{div}(\rho v) = f, \quad (1.14)$$

where the constant $\phi \in (0, 1)$ is the porosity, f is the external mass flow rate.

Combining (1.12) and (1.14), we obtain

$$\phi \rho_t - \kappa \nabla \cdot (K(|\kappa \nabla \rho|) \nabla \rho) = f. \quad (1.15)$$

Then by scaling the time variable in (1.15), we can assume that the multiple factor is 1. Hence (1.15) becomes

$$\rho_t - \nabla \cdot (K(|\nabla \rho|) \nabla \rho) = f. \quad (1.16)$$

This equation is a nonlinear degenerate parabolic equation as the density gradient goes to infinity. For the existence and regularity theory of degenerate parabolic equations, see e.g. [8, 27, 17].

The numerical analysis of the degenerate parabolic equation arising in flow in porous media using mixed finite element approximations was first studied in [1]. Shortly thereafter, Woodward and Dawson in [36] studied the expanded mixed finite element methods for a nonlinear parabolic equation modeling flow into variably saturated porous media. Recently, Galerkin finite element method for a coupled nonlinear degenerate system of advection-diffusion equations were studied in [10, 11, 12, 13]. In their analysis, the Kirchhoff transformation is used to move the nonlinearity from coefficient K to the gradient and thus simplifies the analysis of the equations. This transformation does not applicable for the equation (1.16).

In this paper, we focus on the case of Degree Condition, see (2.3) in the next section, for the following reasons. First, it already covers the most commonly used Forchheimer equations in practice, namely, the two-term, three-term and power laws. Second, it takes advantage of the well-known Poincaré-Sobolev embeddings in our work. Third, it makes clear our ideas and techniques without involving much more complicated technical details in case that the Degree Condition is not met (see [17, 26]). For our degenerate equations, we combine the techniques in [15, 16, 17, 18, 19, 22, 25, 34] and utilize the special structures of the equations to obtain the long-time stability and error estimates for the approximate solution in several norms of interest. Though the error estimates are not optimal order due to the lack of regularity of the solution, these results are obtained with the minimum regularity assumptions.

The paper is organized as follows. In section §2, we introduce notations and some of relevant results. In section §3, we consider the semidiscrete finite element Galerkin approximation and the implicit backward difference time discretization to the initial boundary value problem (3.1). In section §4, we establish many estimates of the energy type norms for the approximate solution $\tilde{\rho}_h$. In Theorems 4.1, 4.4, 4.2, the bounds for approximate solution, its time derivative and gradient vector are established for all time and time $t \rightarrow \infty$. The uniformly large time estimates, asymptotic estimates are in Theorems 4.3, 4.5. In section §5, we analyze two versions of the Galerkin finite element approximations: the continuous Galerkin method and the discrete Galerkin method. Using the monotonicity properties of Forchheimer equation and the boundedness of the solution, the *priori* error estimates for all time, long time, are derived for the solution in L^2 , L^∞ and for the gradient vector in L^{2-a} . The main results are stated and proved in Theorems 5.1–5.3. In section §6, the results of a few numerical experiments using the Lagrange elements of order 1 in the two-dimensions are reported. These results support our theoretical analysis regarding convergence rates.

2. Notations and auxiliary results. Suppose that Ω is an open, bounded subset of \mathbb{R}^d , $d=2,3,\dots$, with boundary Γ smooth. Let $L^2(\Omega)$ be the set of square integrable functions on Ω and $(L^2(\Omega))^d$ the space of d -dimensional vectors with all the components in $L^2(\Omega)$. We denote (\cdot, \cdot) the inner product in either $L^2(\Omega)$ or $(L^2(\Omega))^d$. The notation $\|\cdot\|$ means scalar norm $\|\cdot\|_{L^2(\Omega)}$ or vector norm $\|\cdot\|_{(L^2(\Omega))^d}$ and $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\Omega)}$ represents the standard Lebesgue norm. Notation $\|\cdot\|_{L^p(L^q)} = \|\cdot\|_{L^p(0,T;L^q(\Omega))}$, $1 \leq p, q < \infty$ means the mixed Lebesgue norm while $\|\cdot\|_{L^p(H^q)} = \|\cdot\|_{L^p(0,T;H^q(\Omega))}$, $1 \leq p, q < \infty$ stands for the mixed Sobolev-Lebesgue norm.

For $1 \leq q \leq +\infty$ and m any nonnegative integer, let

$$W^{m,q}(\Omega) = \{u \in L^q(\Omega), D^q u \in L^q(\Omega), |q| \leq m\}$$

denote a Sobolev space endowed with the norm

$$\|u\|_{m,q} = \left(\sum_{|i| \leq m} \|D^i u\|_{L^q(\Omega)}^q \right)^{\frac{1}{q}}.$$

Define $H^m(\Omega) = W^{m,2}(\Omega)$ with the norm $\|\cdot\|_m = \|\cdot\|_{m,2}$.

Throughout this paper, we use short hand notations,

$$\|\rho(t)\| = \|\rho(\cdot, t)\|_{L^2(\Omega)}, \forall t \geq 0 \quad \text{and} \quad \rho^0(\cdot) = \rho(\cdot, 0).$$

Our calculations frequently use the following exponents

$$a = \frac{\alpha_N}{\alpha_N + 1} = \frac{\deg(g)}{\deg(g) + 1}, \quad (2.1)$$

$$\beta = 2 - a, \quad \lambda = \frac{2 - a}{1 - a} = \frac{\beta}{\beta - 1}, \quad \gamma = \frac{a}{2 - a} = \frac{a}{\beta}. \quad (2.2)$$

The letters $C, C_0, C_1, C_2 \dots$ represent for the positive generic constants. Their values depend on exponents, coefficients of polynomial g , the spatial dimension d and domain Ω , independent of the initial and boundary data, size of mesh and time step. These constants may be different from place to place.

Degree Condition: All of the following are equivalent conditions:

$$\deg(g) \leq \frac{4}{d-2}, \quad a \leq \frac{4}{d+2}, \quad 2 \leq \beta^*, \quad 2 - a \geq \frac{2d}{d+2}. \quad (2.3)$$

Here β^* is the Sobolev conjugate of β , given by $\beta^* = \frac{\beta d}{\beta - d}$.

Throughout this paper, we assume the Degree Condition. Whenever this condition is met, the Sobolev space $W^{1,\beta}(\Omega)$ is continuously embedded into $L^2(\Omega)$.

LEMMA 2.1 (cf. [2, 15], Lemma 2.1). *The function $K(\xi)$ has the following properties*

- i. $K : [0, \infty) \rightarrow (0, a_0^{-1}]$ and it decreases in ξ ,
- ii. For any $n \geq 1$, the function $K(\xi)\xi^n$ increasing and $K(\xi)\xi^n \geq 0$
- iii. Type of degeneracy

$$\frac{c_1}{(1 + \xi)^a} \leq K(\xi) \leq \frac{c_2}{(1 + \xi)^a}, \quad (2.4)$$

iv. For all $n \geq 1, \delta > 0$,

$$c_3 \left(\frac{\delta}{1 + \delta} \right)^a (\xi^{n-a} - \delta^{n-a}) \leq K(\xi)\xi^n \leq c_2 \xi^{n-a}, \quad (2.5)$$

In particular, when $n = 2, \delta = 1$

$$2^{-a} c_3 (\xi^{2-a} - 1) \leq K(\xi)\xi^2 \leq c_2 \xi^{2-a}, \quad (2.6)$$

v. Relation with its derivative

$$-aK(\xi) \leq K'(\xi)\xi \leq 0, \quad (2.7)$$

where c_1, c_2, c_3 are positive constants depending on Ω and g .

We define

$$H(\xi) = \int_0^{\xi^2} K(\sqrt{s}) dx, \text{ for } \xi \geq 0. \quad (2.8)$$

The function $H(\xi)$ is compared with ξ and $K(\xi)$ by

$$K(\xi)\xi^2 \leq H(\xi) \leq 2K(\xi)\xi^2. \quad (2.9)$$

For the monotonicity and continuity of the differential operator in (1.16), we have the following results.

LEMMA 2.2 (cf. [15, 2], Lemma 5.2, Lemma III.11). *The following statements hold*

i. *For all $y, y' \in \mathbb{R}^d$,*

$$(K(|y'|)y' - K(|y|)y) \cdot (y' - y) \geq (\beta - 1)K(\max\{|y|, |y'|\})|y' - y|^2. \quad (2.10)$$

ii. *For the vector functions s_1, s_2 , there is a positive constant $c_4(\Omega, d, g)$ such that*

$$(K(|s_1|)s_1 - K(|s_2|)s_2, s_1 - s_2) \geq c_4\omega \|s_1 - s_2\|_{0,\beta}^2, \quad (2.11)$$

where

$$\omega = (1 + \max\{\|s_1\|_{0,\beta}; \|s_2\|_{0,\beta}\})^{-a}.$$

LEMMA 2.3 (cf. [26], Lemma 2.4). *For all vector $y, y' \in \mathbb{R}^d$. There exists a positive constant c_5 depending on polynomial g , the spatial dimension d and domain Ω such that*

$$|K(|y'|)y' - K(|y|)y| \leq c_5|y' - y|. \quad (2.12)$$

The following Poincaré-Sobolev inequality with weight is used in our estimate later.

LEMMA 2.4 (cf. [15] Lemma 2.4). *Let $\xi(x) \geq 0$ be defined on Ω . Then for any function $u(x)$ vanishing on the boundary Γ , there is a positive constant $c_6(\Omega, d, N, g)$ such that*

$$\|u\|_{0,\beta^*}^2 \leq c_6 \left\| K^{\frac{1}{2}}(\xi) \nabla u \right\|^2 \left(1 + \left\| K^{\frac{1}{2}}(\xi) \xi \right\|^2 \right)^\gamma. \quad (2.13)$$

Under degree condition (DC), i.e. $\deg(g) \leq \frac{4}{d-2}$, we have

$$\|u\|^2 \leq c_6 \left\| K^{\frac{1}{2}}(\xi) \nabla u \right\|^2 \left(1 + \left\| K^{\frac{1}{2}}(\xi) \xi \right\|^2 \right)^\gamma. \quad (2.14)$$

DEFINITION 2.5. *Given $f(t)$ defined on an interval $I \subset \mathbb{R}$. A function $F(t)$ is called an (upper) envelop of $f(t)$ on I if $F(t) \geq f(t)$ for all $t \in I$. We denote by $Env(f)$ a continuous, increasing envelop function of $f(t)$.*

We state several Gronwall-type inequalities which are useful in our estimates analysis.

LEMMA 2.6 (cf. [17], Lemma 2.7). *Let $\theta > 0$ and let $y(t) \geq 0, h(t) > 0, f(t) \geq 0$ be continuous functions on $[0, \infty)$ that satisfy*

$$y'(t) + h(t)y(t)^\theta \leq f(t), \quad \text{for all } t > 0.$$

Then

$$y(t) \leq y(0) + \left[Env \left(\frac{f(t)}{h(t)} \right) \right]^{\frac{1}{\theta}}, \quad \text{for all } t \geq 0. \quad (2.15)$$

If $\int_0^\infty h(t)dt = \infty$ then

$$\limsup_{t \rightarrow \infty} y(t) \leq \limsup_{t \rightarrow \infty} \left[\frac{f(t)}{h(t)} \right]^{\frac{1}{\theta}}. \quad (2.16)$$

LEMMA 2.7 (cf. [25] Lemma 2.4). Assume $f \geq 0$, $h, \theta > 0$ and $y(t) \geq 0$ be a continuous function on $[0, \infty)$ satisfying

$$y'(t) + hy(t)^\theta \leq f, \quad \text{for all } t \geq 0$$

then

$$y(t) \leq \max \left\{ y(0), \left(\frac{f}{h} \right)^{1/\theta} \right\}.$$

LEMMA 2.8 (cf. [25] Lemma 2.5, Discrete Gronwall's inequality). Assume $f \geq 0$, $h > 0$, $\theta > 0$, $\Delta t > 0$ and the sequence $\{y_n\}_{n=1}^\infty$ nonnegative satisfying

$$\frac{y_n - y_{n-1}}{\Delta t} + hy_n^\theta \leq f, \quad \text{for all } n = 1, 2, \dots$$

then

$$y_n \leq \max \left\{ y_0, \left(\frac{f}{h} \right)^{1/\theta} \right\}.$$

3. The Galerkin finite element method. We consider the initial boundary value problem associated with (1.16),

$$\begin{cases} \rho_t - \nabla \cdot (K(|\nabla \rho|) \nabla \rho) = f, & \text{in } \Omega \times \mathbb{R}_+, \\ \rho(x, 0) = \rho^0(x), & \text{in } \Omega, \\ \rho(x, t) = \psi(x, t), & \text{on } \Gamma \times \mathbb{R}_+, \end{cases} \quad (3.1)$$

where $\rho^0(x)$ and $\psi(x, t)$ are given initial and boundary data, respectively.

To deal with the non-homogeneous boundary condition, we extend the Dirichlet boundary data from boundary Γ to the whole domain Ω , see [15, 23, 29]. Let $\phi(x, t)$ be such an extension. Let $\tilde{\rho} = \rho - \phi$. Then $\tilde{\rho}(x, t) = 0$ on $\Gamma \times \mathbb{R}_+$. System (3.1) is rewritten as

$$\begin{cases} \tilde{\rho}_t - \nabla \cdot (K(|\nabla \rho|) \nabla \rho) = \tilde{f}, & \text{in } \Omega \times \mathbb{R}_+, \\ \tilde{\rho}(x, 0) = \tilde{\rho}^0(x), & \text{in } \Omega, \\ \tilde{\rho}(x, t) = 0, & \text{on } \Gamma \times \mathbb{R}_+, \end{cases} \quad (3.2)$$

where $\tilde{\rho}^0 = \rho^0(x) - \phi(x, 0)$ and $\tilde{f} = f - \phi_t$.

The variational formulation of (3.1) is defined as the follows: Find $\tilde{\rho} : \mathbb{R}_+ \rightarrow W \equiv H_0^1$ such that

$$(\tilde{\rho}_t, w) + (K(|\nabla \rho|) \nabla \rho, \nabla w) = (\tilde{f}, w), \quad \forall w \in H_0^1(\Omega) \quad (3.3)$$

with $\tilde{\rho}(x, 0) = \tilde{\rho}^0(x)$.

Let $\{\mathcal{T}_h\}_h$ be a family of globally quasiuniform triangulations of Ω with h being the maximum diameter of the element. Let W_h be the space of discontinuous piecewise polynomials of degree $r \geq 0$ over \mathcal{T}_h . It is frequently valuable to decompose the analysis of the convergence of finite element methods by passing through a projection of the solution of the differential problem into the finite element space.

We use the standard L^2 -projection operator, see [7], $\pi : H^1(\Omega) \rightarrow W_h$, satisfying

$$(\pi w, v_h) = (w, v_h), \quad \forall w \in W, v_h \in W_h.$$

This projection has well-known approximation properties, see [6, 24, 4].

i. For all $w \in H^s(\Omega)$, $s \in \{0, 1\}$, there is a positive constant C_0 such that

$$\|\pi w\|_s \leq C_0 \|w\|_s. \quad (3.4)$$

ii. There exists a positive constant C_1 such that

$$\|\pi w - w\|_{0,q} \leq C_1 h^m \|w\|_{m,q} \quad (3.5)$$

for all $w \in W^{m,q}(\Omega)$, $0 \leq m \leq r+1$, $1 \leq q \leq \infty$.

The semidiscrete formulation of (3.3) can read as follows: Find $\tilde{\rho}_h = \rho_h - \pi\phi : \mathbb{R}_+ \rightarrow W_h$ such that

$$(\tilde{\rho}_{h,t}, w_h) + (K(|\nabla \rho_h|) \nabla \rho_h, \nabla w_h) = (\tilde{f}, w_h), \quad \forall w_h \in W_h \quad (3.6)$$

with initial data $\tilde{\rho}_h^0 = \pi \tilde{\rho}^0(x)$.

We use backward Euler for time-difference discretization. Let $\{t_i\}_{i=1}^\infty$ be the uniform partition of \mathbb{R}_+ with $t_i = i \Delta t$, for time step $\Delta t > 0$. We define $\varphi^n = \varphi(\cdot, t_n)$.

The discrete time Galerkin finite element approximation to (3.3) is defined as follows: Find $\tilde{\rho}_h^n \in W_h$, $n = 1, 2, \dots$ such that

$$\left(\frac{\tilde{\rho}_h^n - \tilde{\rho}_h^{n-1}}{\Delta t}, w_h \right) + (K(|\nabla \rho_h^n|) \nabla \rho_h^n, \nabla w_h) = (\tilde{f}^n, w_h), \quad \forall w_h \in W_h. \quad (3.7)$$

The initial data is chosen by $\tilde{\rho}_h^0(x) = \pi \tilde{\rho}^0(x)$.

4. A priori estimate for solutions. We study the equations (3.3), and (3.6) for the density with fixed functions $g(s)$ in (1.7) and (1.8). Therefore, the exponents α_i and coefficients a_i are all fixed, and so are the functions $K(\xi)$, $H(\xi)$ in (1.13), (2.8).

With the properties (2.4), (2.5), (2.7), the monotonicity (2.10), and by classical theory of monotone operators [28, 32, 37], the authors in [17] proved the global existence and uniqueness of the weak solution of the equation (3.3). For the *priori* estimates, we assume that the weak solution is a sufficient regularity in both x and t variables. Hereafter, we only consider solutions $\tilde{\rho}(x; t)$ that satisfy $\tilde{\rho} \in C^2(\bar{\Omega} \times \mathbb{R}_+)$ and $\tilde{\rho}, \nabla \tilde{\rho} \in C(\bar{\Omega} \times \mathbb{R}_+)$.

THEOREM 4.1. *Let $\tilde{\rho}_h$ be a solution of the problem (3.6). Then, there exists a positive constant C such that for all $t > 0$,*

$$\|\tilde{\rho}_h(t)\| \leq C \left(1 + \|\tilde{\rho}^0\| + (Env g(t))^{\frac{1}{p}} \right), \quad (4.1)$$

where

$$g(t) = \|\tilde{f}(t)\|^\lambda + \|\phi(t)\|_1^2. \quad (4.2)$$

Furthermore,

$$\limsup_{t \rightarrow \infty} \|\tilde{\rho}_h(t)\|^2 \leq C \left(1 + \limsup_{t \rightarrow \infty} g(t)\right)^{\frac{2}{\beta}}. \quad (4.3)$$

If

$$\limsup_{t \rightarrow \infty} \|\tilde{f}(t)\| = \limsup_{t \rightarrow \infty} \|\phi(t)\|_1 = 0 \quad (4.4)$$

then

$$\limsup_{t \rightarrow \infty} \|\tilde{\rho}_h(t)\|^2 = 0. \quad (4.5)$$

Proof. Selecting $w_h = \tilde{\rho}_h$ in (3.6), we obtain

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\rho}_h\|^2 + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2 = (K|\nabla \rho_h|) \nabla \rho_h, \nabla \pi \phi + (\tilde{f}, \tilde{\rho}_h). \quad (4.6)$$

By Cauchy's inequality and the upper boundedness of $K(\cdot)$, we have

$$(K|\nabla \rho_h|) \nabla \rho_h, \nabla \pi \phi \leq \frac{1}{4} \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2 + C \|\nabla \pi \phi\|^2. \quad (4.7)$$

Using Hölder's inequality and (2.14) give

$$(\tilde{f}, \tilde{\rho}_h) \leq \|\tilde{f}\| \|\tilde{\rho}_h\| \leq C \|\tilde{f}\| \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \tilde{\rho}_h \right\| \left(1 + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2\right)^{\frac{\gamma}{2}}.$$

Thanks to triangle inequality, $(a+b)^\gamma \leq 2^\gamma (a^\gamma + b^\gamma)$, $\forall a, b \geq 0$, and Young's inequality we find that

$$\begin{aligned} & \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \tilde{\rho}_h \right\| \left(1 + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2\right)^{\frac{\gamma}{2}} \\ & \leq C \left(\left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\| + \|\nabla \pi \phi\| \right) \left(1 + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^\gamma\right) \\ & \leq C \left\{ \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\| + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^{\gamma+1} + \|\nabla \pi \phi\| \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^\gamma + \|\nabla \pi \phi\| \right\} \\ & \leq C \left\{ 1 + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^{\gamma+1} + \|\nabla \pi \phi\|^{\gamma+1} \right\}. \end{aligned}$$

This and Young's inequality applying for $\|\tilde{f}\| \|\nabla \pi \phi\|^{\gamma+1}$ with the exponents $\lambda, \frac{\lambda}{\lambda-1}$ yield

$$\begin{aligned} (\tilde{f}, \tilde{\rho}_h) & \leq C \|\tilde{f}\| \left\{ 1 + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^{\gamma+1} + \|\nabla \pi \phi\|^{\gamma+1} \right\} \\ & \leq C \|\tilde{f}\| + C \|\tilde{f}\|^\lambda + \frac{1}{4} \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2 + C \|\nabla \pi \phi\|^2. \end{aligned} \quad (4.8)$$

Combining (4.7), (4.8) and (4.6) gives

$$\frac{d}{dt} \|\tilde{\rho}_h\|^2 + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2 \leq C \left(\|\nabla \pi \phi\|^2 + \|\tilde{f}\| + \|\tilde{f}\|^\lambda \right). \quad (4.9)$$

We have from (2.5) that

$$c_3 \left(\frac{\delta}{1+\delta} \right)^a \left(\|\nabla \rho_h\|_{0,\beta}^\beta - \delta^\beta \right) \leq \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2.$$

In virtue of the inequality $(a+b)^m \leq 2^{m-1}(a^m + b^m)$, $\forall a, b \geq 0, m \geq 1$,

$$\|\nabla \tilde{\rho}_h\|_{0,\beta}^\beta \leq 2^{\beta-1} \left(\|\nabla \rho_h\|_{0,\beta}^\beta + \|\nabla \pi \phi\|_{0,\beta}^\beta \right).$$

Combining the two above inequalities gives

$$c_3 \left(\frac{\delta}{1+\delta} \right)^a \left(2^{1-\beta} \|\nabla \tilde{\rho}_h\|_{0,\beta}^\beta - \|\nabla \pi \phi\|_{0,\beta}^\beta - \delta^\beta \right) \leq \left\| K^{\frac{1}{2}} (\nabla \rho_h) \nabla \rho_h \right\|^2. \quad (4.10)$$

Under the condition on the degree of the polynomial g , i.e under (DC), using the Poincaré-Sobolev inequality, we obtain

$$\|\tilde{\rho}_h\| \leq C \|\tilde{\rho}_h\|_{0,\beta^*} \leq C_p \|\nabla \tilde{\rho}_h\|_{0,\beta}. \quad (4.11)$$

It follows from (4.9), (4.10) and (4.11) that

$$\begin{aligned} \frac{d}{dt} \|\tilde{\rho}_h\|^2 + c_3 2^{1-\beta} C_p^{-\beta} \left(\frac{\delta}{1+\delta} \right)^a \|\tilde{\rho}_h\|^\beta \\ \leq C \left(\|\tilde{f}\|^\lambda + \|\tilde{f}\| + \|\nabla \pi \phi\|^2 \right) + C \left(\frac{\delta}{1+\delta} \right)^a \|\nabla \pi \phi\|_{0,\beta}^\beta + C \left(\frac{\delta}{1+\delta} \right)^a \delta^\beta \\ \leq C \left(\|\tilde{f}\|^\lambda + \|\tilde{f}\| + \|\nabla \pi \phi\|^2 + \|\nabla \pi \phi\|_{0,\beta}^\beta \right) + C \left(\frac{\delta}{1+\delta} \right)^a \delta^\beta. \end{aligned}$$

According to the Gronwall's inequality in Lemma 2.6 with $\delta = 1$, we find that

$$\begin{aligned} \|\tilde{\rho}_h\|^2 &\leq C \|\tilde{\rho}_h^0\|^2 + C \left[\text{Env} \left(\|\tilde{f}\|^\lambda + \|\tilde{f}\| + \|\nabla \pi \phi\|^2 + \|\nabla \pi \phi\|^\beta + 1 \right) \right]^{\frac{2}{\beta}} \\ &\leq C \|\tilde{\rho}_h^0\|^2 + C \left[\text{Env} \left(\|\tilde{f}\|^\lambda + \|\nabla \pi \phi\|^2 \right) + 1 \right]^{\frac{2}{\beta}}. \end{aligned} \quad (4.12)$$

This and the stability of L^2 - projection (3.4) show that (4.1) holds true.

Due to $\int_0^\infty c_3 2^{1-\beta} C_p^{-\beta} dt = \infty$, it follows from (2.16) that,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|\tilde{\rho}_h\|^2 &\leq C \left[\left(\frac{\delta}{1+\delta} \right)^{-a} \limsup_{t \rightarrow \infty} \left(\|\tilde{f}\|^\lambda + \|\tilde{f}\| + \|\nabla \pi \phi\|^2 + \|\nabla \pi \phi\|^\beta \right) + \delta^\beta \right]^{\frac{2}{\beta}} \\ &\leq C \left[\left(\frac{\delta}{1+\delta} \right)^{-a} \limsup_{t \rightarrow \infty} \left(\|\tilde{f}\|^\lambda + \|\tilde{f}\| + \|\phi\|_1^2 + \|\phi\|_1^\beta \right) + \delta^\beta \right]^{\frac{2}{\beta}}. \end{aligned} \quad (4.13)$$

Therefore, if we choose $\delta = 1$ in the previous inequality then we obtain (4.3).

Under assume (4.4) then

$$\limsup_{t \rightarrow \infty} \left(\|\tilde{f}\|^\lambda + \|\tilde{f}\| + \|\phi\|_1^2 + \|\phi\|_1^\beta \right) = 0.$$

Hence,

$$\limsup_{t \rightarrow \infty} \|\tilde{\rho}_h\|^2 \leq C \delta^2.$$

Letting $\delta \rightarrow 0$, we obtain

$$\limsup_{t \rightarrow \infty} \|\tilde{\rho}_h\|^2 = 0.$$

The proof is complete. \square

Since the equation (3.6) can be interpreted as the finite system of ordinary differential equations in the coefficients of ρ_h with respect to basis of W_h . The stability estimate (4.1) suffices to establish the local existence of $\rho_h(t)$ for all $t \in \mathbb{R}_+$. The uniqueness of the approximation solution comes from the monotonicity of operator, see [15].

Now we derive an estimate for the gradient of approximated solution.

THEOREM 4.2. *Assume $\tilde{\rho}_h$ a solution to the problem (3.6). Then, there exists a positive constant C such that for all $t \geq 0$,*

$$\|\nabla \rho_h(t)\|_{0,\beta}^\beta \leq C \mathcal{A}(t), \quad (4.14)$$

where

$$\mathcal{A}(t) = 1 + \|\rho^0\|_1^2 + \|\phi^0\|^2 + \int_0^t e^{-\frac{1}{2}(t-s)} \left(\|\phi_t(s)\|_1^2 + (\text{Env}g(s))^{\frac{2}{\beta}} \right) ds. \quad (4.15)$$

Furthermore,

$$\limsup_{t \rightarrow \infty} \|\nabla \rho_h(t)\|_{0,\beta}^\beta \leq C \left(1 + \limsup_{t \rightarrow \infty} \|\phi_t(t)\|_1^2 + (\limsup_{t \rightarrow \infty} g(t))^{\frac{2}{\beta}} \right). \quad (4.16)$$

If

$$\limsup_{t \rightarrow \infty} \|\tilde{f}(t)\| = \limsup_{t \rightarrow \infty} \|\phi(t)\|_1 = \limsup_{t \rightarrow \infty} \|\phi_t(t)\|_1 = 0 \quad (4.17)$$

then

$$\limsup_{t \rightarrow \infty} \|\nabla \rho_h(t)\|_{0,\beta} = 0. \quad (4.18)$$

Proof. Choosing $w_h = \tilde{\rho}_{h,t}$ in (3.6) leads to

$$\|\tilde{\rho}_{h,t}\|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} H(x, t) dx = (K(|\nabla \rho_h|) \nabla \rho_h, \nabla \pi \phi_t) + (\tilde{f}, \tilde{\rho}_{h,t}). \quad (4.19)$$

where $H(x, t) = H(|\nabla \rho_h(x, t)|)$ is defined in (2.8).

Adding the two equations (4.19) and (4.6) gives

$$\begin{aligned} \|\tilde{\rho}_{h,t}\|^2 + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2 + \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} H(x, t) dx + \|\tilde{\rho}_h\|^2 \right) \\ = (K|\nabla \rho_h|) \nabla \rho_h, \nabla \pi \phi + \nabla \pi \phi_t + (\tilde{f}, \tilde{\rho}_h + \tilde{\rho}_{h,t}). \end{aligned}$$

Using Cauchy's inequality gives

$$\begin{aligned} \|\tilde{\rho}_{h,t}\|^2 + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2 + \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} H(x, t) dx + \|\tilde{\rho}_h\|^2 \right) \\ \leq \frac{1}{2} \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2 + \frac{1}{2} \|\nabla \pi \phi + \nabla \pi \phi_t\|^2 + 2 \|\tilde{f}\|^2 + \frac{1}{4} \|\tilde{\rho}_h\|^2 + \frac{1}{4} \|\tilde{\rho}_{h,t}\|^2, \end{aligned}$$

which implies

$$\begin{aligned} \frac{3}{4} \|\tilde{\rho}_{h,t}\|^2 + \frac{1}{2} \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2 + \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} H(x, t) dx + \|\tilde{\rho}_h\|^2 \right) \\ \leq \|\nabla \pi \phi\|^2 + \|\nabla \pi \phi_t\|^2 + 2 \|\tilde{f}\|^2 + \frac{1}{4} \|\tilde{\rho}_h\|^2. \quad (4.20) \end{aligned}$$

Now using (2.9), we find that

$$\left\| K^{\frac{1}{2}}(|\nabla \rho_h|)\nabla \rho_h \right\|^2 \geq \frac{1}{2} \int_{\Omega} H(x, t) dx.$$

This and (4.20) show that

$$\begin{aligned} \frac{3}{4} \|\bar{\rho}_{h,t}\|^2 + \frac{1}{4} \int_{\Omega} H(x, t) dx + \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} H(x, t) dx + \|\bar{\rho}_h\|^2 \right) \\ \leq \|\nabla \pi \phi\|^2 + \|\nabla \pi \phi_t\|^2 + 2 \|\tilde{f}\|^2 + \frac{1}{4} \|\bar{\rho}_h\|^2. \end{aligned} \quad (4.21)$$

Note that $\frac{1}{2} \frac{d}{dt} \|\bar{\rho}_h\|^2 = (\bar{\rho}_h, \bar{\rho}_{h,t})$, again using Cauchy's inequality leads to

$$\begin{aligned} \frac{3}{4} \|\bar{\rho}_{h,t}\|^2 + \frac{1}{4} \int_{\Omega} H(x, t) dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} H(x, t) dx \\ \leq \|\nabla \pi \phi\|^2 + \|\nabla \pi \phi_t\|^2 + 2 \|\tilde{f}\|^2 + \frac{1}{4} \|\bar{\rho}_h\|^2 + \frac{1}{2} \|\bar{\rho}_h\|^2 + \frac{1}{2} \|\bar{\rho}_{h,t}\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{4} \|\bar{\rho}_{h,t}\|^2 + \frac{1}{4} \int_{\Omega} H(x, t) dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} H(x, t) dx \\ \leq \|\nabla \pi \phi\|^2 + \|\nabla \pi \phi_t\|^2 + 2 \|\tilde{f}\|^2 + \frac{3}{4} \|\bar{\rho}_h\|^2. \end{aligned} \quad (4.22)$$

This gives

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} H(x, t) dx + \frac{1}{2} \left(\|\bar{\rho}_{h,t}\|^2 + \int_{\Omega} H(x, t) dx \right) \leq 4 \left(\|\nabla \pi \phi\|^2 + \|\nabla \pi \phi_t\|^2 + \|\tilde{f}\|^2 + \|\bar{\rho}_h\|^2 \right) \\ \leq C \left(\|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 + \|\bar{\rho}_h\|^2 \right). \end{aligned} \quad (4.23)$$

Dropping the nonnegative term on the left hand side in (4.23), using (4.1), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} H(x, t) dx + \frac{1}{2} \int_{\Omega} H(x, t) dx \leq C \left(\|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 + \|\bar{\rho}_h\|^2 \right) \\ \leq C \left(\|\bar{\rho}^0\|^2 + \|\phi_t\|_1^2 + (Env(g(t)))^{\frac{2}{\beta}} + 1 \right). \end{aligned} \quad (4.24)$$

The last inequality is obtained by using the Young's inequality then absorbing $\|\tilde{f}\|^\lambda$ and $\|\phi\|_1^2$ to $Env(g(t))^{\frac{2}{\beta}}$.

It is followed by applying Gronwall's inequality to (4.24) that

$$\begin{aligned} \int_{\Omega} H(x, t) dx \leq e^{-\frac{1}{2}t} \int_{\Omega} H(x, 0) dx \\ + C \int_0^t e^{-\frac{1}{2}(t-s)} \left(\|\bar{\rho}^0\|^2 + \|\phi_t\|_1^2 + (Env(g(t)))^{\frac{2}{\beta}} + 1 \right) ds. \end{aligned} \quad (4.25)$$

Due to (2.9) and (2.5),

$$\begin{aligned} \|\nabla \rho_h\|_{0,\beta}^\beta &\leq C e^{-\frac{1}{2}t} \|\nabla \rho_h^0\|_{0,\beta}^\beta \\ &+ C \int_0^t e^{-\frac{1}{2}(t-s)} \left(\|\bar{\rho}^0\|^2 + \|\phi_t\|_1^2 + (Env(g(t)))^{\frac{2}{\beta}} + 1 \right) ds + C \\ &\leq C + C \|\rho^0\|_1^2 + C \|\bar{\rho}^0\|^2 + C \int_0^t e^{-\frac{1}{2}(t-s)} \left(\|\phi_t\|_1^2 + (Env(g(t)))^{\frac{2}{\beta}} \right) ds. \end{aligned}$$

This proves (4.14).

Dropping the nonnegative term on the left hand side of (4.23) and using (2.16) to (4.23), we find that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_{\Omega} H(x, t) dx &\leq C \left[\limsup_{t \rightarrow \infty} \left(\|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 \right) + \limsup_{t \rightarrow \infty} \|\bar{\rho}_h\|^2 \right] \\ &\leq C \left[\limsup_{t \rightarrow \infty} \left(1 + g(t) + \|\phi_t\|_1^2 \right) + \left(\limsup_{t \rightarrow \infty} g(t) \right)^{\frac{2}{\beta}} + 1 \right] \\ &\leq C \left[\limsup_{t \rightarrow \infty} \|\phi_t\|_1^2 + \left(\limsup_{t \rightarrow \infty} g(t) \right)^{\frac{2}{\beta}} + 1 \right]. \end{aligned} \quad (4.26)$$

From (2.5) and (2.9) we have

$$c_3 \left(\frac{\delta}{1+\delta} \right)^a \left(\|\nabla \rho_h\|_{0,\beta}^\beta - \delta^\beta \right) \leq \left\| K^{\frac{1}{2}} (|\nabla \rho_h|) \nabla \rho_h \right\|^2 \leq \int_{\Omega} H(x, t). \quad (4.27)$$

With $\delta = 1$, combining (4.26) and (4.27) leads to (4.16).

Assume (4.17) then (4.26) and (4.27) lead to

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|\nabla \rho_h\|_{0,\beta}^\beta &\leq C \left(\frac{\delta}{1+\delta} \right)^{-a} \left[\limsup_{t \rightarrow \infty} \left(\|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 \right) + \limsup_{t \rightarrow \infty} \|\bar{\rho}_h\|^2 \right] + C\delta^\beta \\ &= C\delta^\beta \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \end{aligned}$$

We complete the proof. \square

The result (4.14) in Theorem 4.2 provides an estimate for the gradient of density at the given time $t = T$, which includes information on boundary data for all time $t \leq T$. When T is large, it needs to be expressed mainly in terms of the boundary data on the interval $[T-1, T]$, uniformly for all T .

THEOREM 4.3. *Assume $\bar{\rho}_h$ a solution to the problem (3.6). The following inequalities hold uniformly*

i. *For all $t \geq 1$,*

$$\int_{t-1}^t \|\nabla \rho_h(\tau)\|_{0,\beta}^\beta d\tau \leq C \left(1 + \|\bar{\rho}_h(t-1)\|^2 + \int_{t-1}^t \left(\|\phi(\tau)\|_1^2 + \|\tilde{f}(\tau)\|^\lambda \right) d\tau \right); \quad (4.28)$$

ii. *For all $t \geq 1$,*

$$\begin{aligned} \int_{t-\frac{1}{2}}^t \|\bar{\rho}_{h,t}(\tau)\|^2 d\tau + \|\nabla \rho_h(t)\|_{0,\beta}^\beta &\leq C \|\bar{\rho}_h(t-1)\|^2 \\ &\quad + C \left(1 + \int_{t-1}^t \left(\|\phi_t(\tau)\|_1^2 + \|\tilde{f}(\tau)\|^\lambda \right) d\tau \right); \end{aligned} \quad (4.29)$$

iii. *For all $t \geq 1$,*

$$\|\nabla \rho_h(t)\|_{0,\beta}^\beta \leq C \left(1 + \|\bar{\rho}^0\|^2 + (Env g(t))^{\frac{2}{\beta}} + \int_{t-1}^t \|\phi_t(\tau)\|_1^2 d\tau \right). \quad (4.30)$$

Proof. Integrating (4.9) in time from $t-1$ to t , we obtain

$$\|\bar{\rho}_h(t)\|^2 + \int_{t-1}^t \left\| K^{\frac{1}{2}} (|\nabla \rho_h|) \nabla \rho_h \right\|^2 d\tau \leq \|\bar{\rho}_h(t-1)\|^2 + C \int_{t-1}^t \left(\|\nabla \pi \phi\|^2 + \|\tilde{f}\| + \|\tilde{f}\|^\lambda \right) d\tau. \quad (4.31)$$

Neglecting nonnegative term on the left hand side of (4.31) shows that

$$\int_{t-1}^t \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2 d\tau \leq \|\tilde{\rho}_h(t-1)\|^2 + C \int_{t-1}^t \left(\|\nabla \pi \phi\|^2 + \|\tilde{f}\| + \|\tilde{f}\|^\lambda \right) d\tau.$$

Using (2.6) and Young's inequality, we obtain (4.28).

Following by applying Cauchy's inequality to (4.19) that

$$\|\tilde{\rho}_{h,t}\|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} H(x, t) dx \leq \frac{1}{2} \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_h \right\|^2 + \frac{1}{2} \left(\|\nabla \pi \phi_t\|^2 + \|\tilde{f}\|^2 + \|\tilde{\rho}_{h,t}\|^2 \right). \quad (4.32)$$

In virtue of (2.9), we find that

$$\|\tilde{\rho}_{h,t}\|^2 + \frac{d}{dt} \int_{\Omega} H(x, t) dx \leq \int_{\Omega} H(x, t) dx + \|\nabla \pi \phi_t\|^2 + \|\tilde{f}\|^2. \quad (4.33)$$

Integrating (4.33) in τ from s to t where $s \in [t-1, t]$, we have

$$\begin{aligned} \int_s^t \|\tilde{\rho}_{h,t}\|^2 d\tau + \int_{\Omega} H(x, t) dx \\ \leq \int_{\Omega} H(x, s) dx + \int_s^t \int_{\Omega} H(x, s) dx d\tau + \int_s^t \left(\|\nabla \pi \phi_t\|^2 + \|\tilde{f}\|^2 \right) d\tau \\ \leq \int_{\Omega} H(x, s) dx + \int_{t-1}^t \int_{\Omega} H(x, t) dx d\tau + \int_{t-1}^t \left(\|\nabla \pi \phi_t\|^2 + \|\tilde{f}\|^2 \right) d\tau. \end{aligned} \quad (4.34)$$

Then integrating (4.34) in s from $t-1$ to t gives

$$\int_{t-1}^t \int_s^t \|\tilde{\rho}_{h,t}\|^2 d\tau ds + \int_{\Omega} H(x, t) dx \leq 2 \int_{t-1}^t \int_{\Omega} H(x, \tau) dx d\tau + \int_{t-1}^t \left(\|\nabla \pi \phi_t\|^2 + \|\tilde{f}\|^2 \right) d\tau. \quad (4.35)$$

We bound the right hand side in (4.35) using (2.9), (4.28) and Young's inequality to obtain

$$\int_{t-1}^t \int_s^t \|\tilde{\rho}_{h,t}\|^2 d\tau ds + \int_{\Omega} H(x, t) dx \leq 2 \|\tilde{\rho}_h(t-1)\|^2 + C \left(1 + \int_{t-1}^t (\|\phi_t\|_1^2 + \|\tilde{f}\|^\lambda) d\tau \right). \quad (4.36)$$

The first term of (4.36) is bounded by

$$\int_{t-1}^t \int_s^t \|\tilde{\rho}_{h,t}\|^2 d\tau ds \geq \int_{t-1}^t \int_{t-\frac{1}{2}}^t \|\tilde{\rho}_{h,t}\|^2 d\tau ds \geq \frac{1}{2} \int_{t-\frac{1}{2}}^t \|\tilde{\rho}_{h,t}\|^2 d\tau. \quad (4.37)$$

Combining (4.36), (4.37) and using (2.6) we obtain (4.29).

The inequality (4.30) follows by using (4.1) to bound the first term of the right hand side in (4.29). \square

Now we prove the time derivative of pressure is bounded.

THEOREM 4.4. *Let $0 < t_0 < 1$, assume $\tilde{\rho}_h$ solves the semidiscrete problem (3.6). Then, there exists a positive constant C such that for all $t \geq t_0$,*

$$\|\tilde{\rho}_{h,t}(t)\|^2 \leq C \mathcal{B}(t). \quad (4.38)$$

where

$$\begin{aligned} \mathcal{B}(t) = t_0^{-1} e^{-\frac{1}{4}(t-t_0)} \left\{ 1 + \|\rho^0\|_1^2 + \|\phi^0\|^2 + \int_0^{t_0} (\|\phi_t(s)\|_1^2 + (\text{Env } g(s))^{\frac{2}{p}}) ds \right\} \\ + \int_0^t e^{-\frac{1}{4}(t-s)} \left(1 + \|\tilde{\rho}^0\|^2 + \|\phi_t(s)\|_1^2 + \|\tilde{f}_t(s)\|^2 + (\text{Env } g(s))^{\frac{2}{p}} \right) ds. \end{aligned} \quad (4.39)$$

Furthermore,

$$\limsup_{t \rightarrow \infty} \|\tilde{\rho}_{h,t}(t)\|^2 \leq C \left(1 + (\limsup_{t \rightarrow \infty} g(t))^{\frac{2}{\beta}} + \limsup_{t \rightarrow \infty} \|\phi_t(t)\|_1^2 + \limsup_{t \rightarrow \infty} \|\tilde{f}_t(t)\|^2 \right). \quad (4.40)$$

Proof. Differentiating (3.6) with respect t yields that

$$(\tilde{\rho}_{h,tt}, w_h) + (K(|\nabla \rho_h|) \nabla \rho_{h,t}, \nabla w_h) = - \left(K'(|\nabla \rho_h|) \frac{\nabla \rho_h \cdot \nabla \rho_{h,t}}{|\nabla \rho_h|} \nabla \rho_h, \nabla w_h \right) + (\tilde{f}_t, w_h).$$

Then choosing $w_h = \tilde{\rho}_{h,t}$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{\rho}_{h,t}\|^2 + \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_{h,t} \right\|^2 &= (K(|\nabla \rho_h|) \nabla \rho_{h,t}, \nabla \pi \phi_t) \\ &- \left(K'(|\nabla \rho_h|) \frac{\nabla \rho_h \cdot \nabla \rho_{h,t}}{|\nabla \rho_h|} \nabla \rho_h, \nabla \rho_{h,t} \right) + \left(K'(|\nabla \rho_h|) \frac{\nabla \rho_h \cdot \nabla \rho_{h,t}}{|\nabla \rho_h|} \nabla \rho_h, \nabla \pi \phi_t \right) + (\tilde{f}_t, \tilde{\rho}_{h,t}). \end{aligned}$$

The Cauchy inequality and the upper boundedness of $K(\cdot)$ give

$$|(K(|\nabla \rho_h|) \nabla \rho_{h,t}, \nabla \pi \phi_t)| \leq \frac{1-a}{2} \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_{h,t} \right\|^2 + C \|\nabla \pi \phi_t\|^2. \quad (4.41)$$

Following from (2.7) that

$$\left| - \left(K'(|\nabla \rho_h|) \frac{\nabla \rho_h \cdot \nabla \rho_{h,t}}{|\nabla \rho_h|} \nabla \rho_h, \nabla \rho_{h,t} \right) \right| \leq a \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_{h,t} \right\|^2. \quad (4.42)$$

Combining (2.7), Cauchy's inequality and the upper boundedness of $K(\cdot)$ gives

$$\begin{aligned} \left| \left(K'(|\nabla \rho_h|) \frac{\nabla \rho_h \cdot \nabla \rho_{h,t}}{|\nabla \rho_h|} \nabla \rho_h, \nabla \pi \phi_t \right) \right| &\leq a \int_{\Omega} K(|\nabla \rho_h|) |\nabla \rho_{h,t}| |\nabla \pi \phi_t| dx \\ &\leq \frac{1-a}{2} \left\| K^{\frac{1}{2}}(|\nabla \rho_h|) \nabla \rho_{h,t} \right\|^2 + C \|\nabla \pi \phi_t\|^2. \end{aligned} \quad (4.43)$$

Using Cauchy's inequality,

$$|(\tilde{f}_t, \tilde{\rho}_{h,t})| \leq \frac{1}{8} \|\tilde{\rho}_{h,t}\|^2 + C \|\tilde{f}_t\|^2. \quad (4.44)$$

Above estimates lead to

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\rho}_{h,t}\|^2 \leq C \|\nabla \pi \phi_t\|^2 + \frac{1}{8} \|\tilde{\rho}_{h,t}\|^2 + C \|\tilde{f}_t\|^2. \quad (4.45)$$

Combining (4.45) and (4.22), we obtain

$$\begin{aligned} \frac{1}{4} \left(\|\tilde{\rho}_{h,t}\|^2 + \int_{\Omega} H(x, t) dx \right) + \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} H(x, t) dx + \|\tilde{\rho}_{h,t}\|^2 \right) \\ \leq C \left(\|\nabla \pi \phi_t\|^2 + \|\tilde{f}_t\|^2 + \|\tilde{\rho}_h\|^2 + \|\nabla \pi \phi_t\|^2 + \|\tilde{f}_t\|^2 \right) + \frac{1}{8} \|\tilde{\rho}_{h,t}\|^2. \end{aligned} \quad (4.46)$$

Define

$$\mathcal{E}(t) = \|\tilde{\rho}_{h,t}\|^2 + \int_{\Omega} H(x, t) dx.$$

Using (4.1), we obtain

$$\frac{d}{dt}\mathcal{E}(t) + \frac{1}{4}\mathcal{E}(t) \leq C \left(\|\tilde{\rho}_h\|^2 + \|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 + \|\tilde{f}_t\|^2 \right). \quad (4.47)$$

For any $t \geq t_0 \geq t' > 0$, integrating (4.47) from t' to t , we find that

$$\mathcal{E}(t) \leq e^{-\frac{1}{4}(t-t')} \mathcal{E}(t') + C \int_0^t e^{-\frac{1}{4}(t-s)} \left[\|\tilde{\rho}_h\|^2 + \|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 + \|\tilde{f}_t\|^2 \right] ds. \quad (4.48)$$

Integrating in t' from 0 to t_0 gives

$$t_0 \mathcal{E}(t) \leq e^{-\frac{1}{4}(t-t_0)} \int_0^{t_0} \mathcal{E}(t') dt' + t_0 C \int_0^t e^{-\frac{1}{4}(t-s)} \left[\|\tilde{\rho}_h\|^2 + \|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 + \|\tilde{f}_t\|^2 \right] ds.$$

Integrating (4.23) from 0 to t_0 we obtain

$$\int_0^{t_0} \mathcal{E}(t') dt' \leq \int_{\Omega} H(x, 0) dx + \int_0^{t_0} \left[\|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 + \|\tilde{\rho}_h\|^2 \right] ds. \quad (4.49)$$

Therefore,

$$\begin{aligned} \mathcal{E}(t) &\leq t_0^{-1} e^{-\frac{1}{4}(t-t_0)} \left\{ \int_{\Omega} H(x, 0) dx + \int_0^{t_0} \left[\|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 + \|\tilde{\rho}_h\|^2 \right] ds \right\} \\ &\quad + C \int_0^t e^{-\frac{1}{4}(t-s)} \left[\|\tilde{\rho}_h\|^2 + \|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 + \|\tilde{f}_t\|^2 \right] ds. \end{aligned} \quad (4.50)$$

Then the inequality (4.38) follows from (4.50), (4.1), (2.6) and (2.9).

Now applying Gronwall's inequality in (2.16) for (4.47) yields

$$\begin{aligned} \limsup_{t \rightarrow \infty} \mathcal{E}(t) &\leq C \limsup_{t \rightarrow \infty} \|\tilde{\rho}_h\|^2 + C \limsup_{t \rightarrow \infty} \left(\|\phi\|_1^2 + \|\phi_t\|_1^2 + \|\tilde{f}\|^2 + \|\tilde{f}_t\|^2 \right) \\ &\leq C \left(\left(\limsup_{t \rightarrow \infty} g(t) \right)^{\frac{2}{p}} + \limsup_{t \rightarrow \infty} \|\phi_t\|_1^2 + \limsup_{t \rightarrow \infty} \|\tilde{f}_t\|^2 + 1 \right). \end{aligned}$$

The last inequality is followed by using (4.3). Then the proof is complete. \square

Next, we derive a uniformly bound for $\tilde{\rho}_{h,t}$.

THEOREM 4.5. *Suppose $\tilde{\rho}_h$ solves the semidiscrete problem (3.6). Then, there exists a positive constant C such that for all $t \geq 1$,*

$$\|\tilde{\rho}_{h,t}(t)\|^2 \leq C \left\{ 1 + \|\tilde{\rho}^0\|^2 + (Env \ g(t))^{\frac{2}{p}} + \int_{t-1}^t \left(\|\phi_t(s)\|_1^2 + \|\tilde{f}_t(s)\|^2 \right) ds \right\}. \quad (4.51)$$

Proof. Integrating (4.45) in time from s to t where $t - \frac{1}{2} \leq s \leq t$, we have

$$\begin{aligned} \|\tilde{\rho}_{h,t}(t)\|^2 &\leq \|\tilde{\rho}_{h,t}(s)\|^2 + \frac{1}{4} \int_s^t \|\tilde{\rho}_{h,t}(\tau)\|^2 d\tau + C \int_s^t (\|\nabla \pi \phi_t\|^2 + \|\tilde{f}_t\|^2) d\tau \\ &\leq \|\tilde{\rho}_{h,t}(s)\|^2 + \frac{1}{4} \int_{t-\frac{1}{2}}^t \|\tilde{\rho}_{h,t}(\tau)\|^2 d\tau + C \int_{t-1}^t (\|\nabla \pi \phi_t\|^2 + \|\tilde{f}_t\|^2) d\tau. \end{aligned} \quad (4.52)$$

Now integrating (4.52) in s from $t - \frac{1}{2}$ to t , we have

$$\|\tilde{\rho}_{h,t}(t)\|^2 \leq \frac{5}{2} \int_{t-\frac{1}{2}}^t \|\tilde{\rho}_{h,t}(s)\|^2 ds + C \int_{t-1}^t (\|\nabla \pi \phi_t\|^2 + \|\tilde{f}_t\|^2) d\tau. \quad (4.53)$$

This and (4.29) yield

$$\|\tilde{\rho}_{h,t}(t)\|^2 \leq C \|\tilde{\rho}_h(t-1)\|^2 + C \left(1 + \int_{t-1}^t \left(\|\phi_t\|_1^2 + \|\tilde{f}\|^\lambda + \|\tilde{f}_t\|^2\right) d\tau\right). \quad (4.54)$$

We use (4.1) to estimate the first term on the right hand side of (4.54). Then (4.51) follows. \square

The proof of the following stability results applied to the problem (3.3) is similar to the proofs in Theorem 4.1–4.5. We omit for brevity.

THEOREM 4.6. *Let $\tilde{\rho}$ be a solution of the problem (3.3). Then, there exists a positive constant C such that*

i. For all $t > 0$,

$$\|\tilde{\rho}(t)\| \leq C \left(1 + \|\rho^0\| + (Env g(t))^{\frac{1}{\beta}}\right); \quad (4.55)$$

$$\limsup_{t \rightarrow \infty} \|\tilde{\rho}(t)\|^2 \leq C \left(1 + \limsup_{t \rightarrow \infty} g(t)\right)^{\frac{2}{\beta}}. \quad (4.56)$$

$$\text{If } \limsup_{t \rightarrow \infty} \|\tilde{f}(t)\| = \limsup_{t \rightarrow \infty} \|\phi(t)\|_1 = 0 \text{ then } \limsup_{t \rightarrow \infty} \|\tilde{\rho}(t)\|^2 = 0. \quad (4.57)$$

ii. For all $t > 0$,

$$\|\nabla \rho(t)\|_{0,\beta}^\beta \leq C \mathcal{A}(t), \text{ where } \mathcal{A}(t) \text{ is defined in (4.15)}. \quad (4.58)$$

$$\limsup_{t \rightarrow \infty} \|\nabla \rho(t)\|_{0,\beta}^\beta \leq C \left(1 + \limsup_{t \rightarrow \infty} \|\phi_t(t)\|_1^2 + (\limsup_{t \rightarrow \infty} g(t))^{\frac{2}{\beta}}\right). \quad (4.59)$$

$$\text{If } \limsup_{t \rightarrow \infty} \{\|\tilde{f}(t)\|, \|\phi(t)\|_1, \|\phi_t(t)\|_1\} = 0 \text{ then } \limsup_{t \rightarrow \infty} \|\nabla \rho(t)\|_{0,\beta} = 0. \quad (4.60)$$

iii. Let $0 < t_0 < 1$, for all $t \geq t_0$,

$$\|\tilde{\rho}_t(t)\|^2 \leq C \mathcal{B}(t), \text{ where } \mathcal{B}(t) \text{ is defined in (4.39)}. \quad (4.61)$$

$$\limsup_{t \rightarrow \infty} \|\tilde{\rho}_t(t)\|^2 \leq C \left(1 + (\limsup_{t \rightarrow \infty} g(t))^{\frac{2}{\beta}} + \limsup_{t \rightarrow \infty} \|\phi_t(t)\|_1^2 + \limsup_{t \rightarrow \infty} \|\tilde{f}_t(t)\|^2\right), \quad (4.62)$$

iv. For all $t \geq 1$,

$$\|\nabla \rho(t)\|_{0,\beta}^\beta \leq C \left(1 + \|\tilde{\rho}^0\|^2 + (Env g(t))^{\frac{2}{\beta}} + \int_{t-1}^t \|\phi_t(\tau)\|_1^2 d\tau\right). \quad (4.63)$$

$$\|\tilde{\rho}_t(t)\|^2 \leq C \left\{1 + \|\tilde{\rho}^0\|^2 + (Env g(t))^{\frac{2}{\beta}} + \int_{t-1}^t \left(\|\phi_t(s)\|_1^2 + \|\tilde{f}_t(s)\|^2\right) ds\right\}. \quad (4.64)$$

5. Error estimates. In this section, we will establish the error estimates between analytical solution and approximation solution in several norms. Let

$$\mathcal{F}(t) = \begin{cases} 1 + \|\rho^0\|_1^2 + \|\phi^0\|^2 + \int_0^t e^{-\frac{1}{2}(t-s)} \left(\|\phi_t(s)\|_1^2 + (Env g(s))^{\frac{2}{\beta}}\right), & \text{if } 0 \leq t \leq 1, \\ 1 + \|\tilde{\rho}^0\|^2 + (Env g(t))^{\frac{2}{\beta}} + \int_{t-1}^t \|\phi_t(\tau)\|_1^2 d\tau, & \text{if } t \geq 1, \end{cases} \quad (5.1)$$

and

$$\mathcal{K} = 1 + \limsup_{t \rightarrow \infty} \|\phi_t(t)\|_1^2 + (\limsup_{t \rightarrow \infty} g(t))^{\frac{2}{\beta}}, \quad \mathcal{L} = \mathcal{K} + \limsup_{t \rightarrow \infty} \|\tilde{f}_t(t)\|^2. \quad (5.2)$$

Define

$$\Lambda(t) = 1 + \|\nabla \rho(t)\|_{0,\beta}^\beta + \|\nabla \rho_h(t)\|_{0,\beta}^\beta. \quad (5.3)$$

According to Theorem 4.2, 4.3 and Theorem 4.6

$$\Lambda(t) \leq C\mathcal{F}(t), \quad \text{and} \quad \limsup_{t \rightarrow \infty} \Lambda(t) \leq \mathcal{K}.$$

Let

$$\mathcal{G}(t) = \begin{cases} t_0^{-1} \left\{ 1 + \|\rho^0\|_1^2 + \|\phi^0\|^2 + \int_0^{t_0} (\|\phi_t(s)\|_1^2 + (Env g(s))^{\frac{2}{\beta}}) ds \right\} \\ + \int_0^t e^{-\frac{1}{4}(t-s)} \left(1 + \|\bar{\rho}^0\|^2 + \|\phi_t(s)\|_1^2 + \|\tilde{f}_t(s)\|^2 + (Env g(s))^{\frac{2}{\beta}} \right) ds, & \text{if } 0 < t_0 \leq t \leq 1, \\ 1 + \|\bar{\rho}^0\|^2 + (Env g(t))^{\frac{2}{\beta}} + \int_{t-1}^t (\|\phi_t(s)\|_1^2 + \|\tilde{f}_t(s)\|^2), & \text{if } t \geq 1. \end{cases} \quad (5.4)$$

We have from Theorem 4.4, 4.5 and Theorem 4.6 that

$$\|\bar{\rho}_t(t)\| + \|\bar{\rho}_{h,t}(t)\| \leq C\sqrt{\mathcal{G}(t)} \quad \text{and} \quad \limsup_{t \rightarrow \infty} (\|\bar{\rho}_t(t)\| + \|\bar{\rho}_{h,t}(t)\|) \leq C\mathcal{L}. \quad (5.5)$$

5.1. Error estimate for continuous Galerkin method. We will find the error bounds in the semidiscrete method by comparing the computed solution to the projections of the exact solutions. To do this, we restrict the test functions in (3.3) to the finite dimensional spaces. Let

$$\chi = \bar{\rho} - \bar{\rho}_h = (\bar{\rho} - \pi\bar{\rho}) - (\bar{\rho}_h - \pi\bar{\rho}) \stackrel{\text{def}}{=} \vartheta - \theta_h, \quad \text{and} \quad \varphi \stackrel{\text{def}}{=} \phi - \pi\phi. \quad (5.6)$$

THEOREM 5.1. *Let $1 \leq k \leq r+1$, $\bar{\rho}, \bar{\rho}_h$ be solutions to (3.3) and (3.6) respectively. Assume that $\bar{\rho} \in L^\infty(\mathbb{R}_+, H^k(\Omega))$, $\bar{\rho}_t \in L^2(\mathbb{R}_+, H^k(\Omega))$. Then, there exists a constant positive constant C independent of h such that for all $t > 0$,*

$$\|(\bar{\rho} - \bar{\rho}_h)(t)\|^2 \leq Ch^{2k} \|\bar{\rho}(t)\|_k^2 + Ch^{k-1} \int_0^t e^{-2^{-a} \int_s^t \Lambda(\tau)^{-1} d\tau} \mathcal{F}(s) \mathcal{H}(s) ds, \quad (5.7)$$

where $\mathcal{F}(t)$ is defined as (5.1), and

$$\mathcal{H}(t) = \|\bar{\rho}_t(t)\|_k^2 + \|\bar{\rho}(t)\|_{k,\beta}^2 + \|\bar{\rho}_h(t)\|_k^2 + \|\phi(t)\|_{k,\beta}^2 + \|\phi(t)\|_{k,\beta}. \quad (5.8)$$

Furthermore, if $\int_0^\infty \Lambda^{-1}(t) dt = \infty$ then

$$\limsup_{t \rightarrow \infty} \|(\bar{\rho} - \bar{\rho}_h)(t)\|^2 \leq Ch^{k-1} \mathcal{K}^2 \limsup_{t \rightarrow \infty} \mathcal{H}(t). \quad (5.9)$$

Proof. From (3.3) and (3.6), we find the error equation

$$(\bar{\rho}_t - \bar{\rho}_{h,t}, w_h) + (K(|\nabla \rho|)\nabla \rho - K(|\nabla \rho_h|)\nabla \rho_h, \nabla w_h) = 0, \quad \forall w_h \in W_h. \quad (5.10)$$

Taking $w_h = \theta_h$, we obtain

$$(\vartheta_t - \theta_{h,t}, \theta_h) + (K(|\nabla \rho|)\nabla \rho - K(|\nabla \rho_h|)\nabla \rho_h, \nabla \theta_h) = 0. \quad (5.11)$$

We rewrite the equation (5.11) as form

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta_h\|^2 + (K(|\nabla \rho|) \nabla \rho - K(|\nabla \rho_h|) \nabla \rho_h, \nabla(\rho - \rho_h)) \\ = (\vartheta_t, \theta_h) + (K(|\nabla \rho|) \nabla \rho - K(|\nabla \rho_h|) \nabla \rho_h, \nabla \vartheta + \nabla \varphi). \end{aligned}$$

Thanks to (2.11),

$$(K(|\nabla \rho|) \nabla \rho - K(|\nabla \rho_h|) \nabla \rho_h, \nabla(\rho - \rho_h)) \geq c_4 \omega \|\nabla(\rho - \rho_h)\|_{0,\beta}^2. \quad (5.12)$$

Using (2.3), Hölder's and Young's inequality, we have

$$\begin{aligned} (K(|\nabla \rho|) \nabla \rho - K(|\nabla \rho_h|) \nabla \rho_h, \nabla \vartheta + \nabla \varphi) &\leq C(|\nabla \rho|^{\beta-1} + |\nabla \rho_h|^{\beta-1}, |\nabla \vartheta + \nabla \varphi|) \\ &\leq C \left(\|\nabla \rho\|_{0,\beta}^{\beta-1} + \|\nabla \rho_h\|_{0,\beta}^{\beta-1} \right) \|\nabla \vartheta + \nabla \varphi\|_{0,\beta} \\ &\leq C \Lambda(t) \left(\|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta} \right). \end{aligned} \quad (5.13)$$

Using Young's inequality, for $\varepsilon > 0$

$$(\vartheta_t, \theta_h) \leq C \omega^{-1} \varepsilon^{-1} \|\vartheta_t\|^2 + \varepsilon \omega \|\theta_h\|^2. \quad (5.14)$$

Combining (5.12), (5.13) and (5.14) gives

$$\frac{1}{2} \frac{d}{dt} \|\theta_h\|^2 + c_4 \omega \|\nabla(\rho - \rho_h)\|_{0,\beta}^2 \leq C \omega^{-1} \varepsilon^{-1} \|\vartheta_t\|^2 + \varepsilon \omega \|\theta_h\|^2 + \Lambda(t) \left(\|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta} \right).$$

By Poincaré-Sobolev inequality $\|u\| \leq C_p \|\nabla u\|_{0,\beta}$ for all $u \in H_0^1(\Omega)$,

$$\begin{aligned} \|\nabla(\rho - \rho_h)\|_{0,\beta}^2 &\geq \frac{1}{2} \|\nabla(\bar{\rho} - \bar{\rho}_h)\|_{0,\beta}^2 - \|\nabla \varphi\|_{0,\beta}^2 \\ &\geq \frac{1}{2C_p^2} \|\bar{\rho} - \bar{\rho}_h\|^2 - \|\nabla \varphi\|_{0,\beta}^2 \\ &\geq \frac{1}{4C_p^2} \|\theta_h\|^2 - \frac{1}{2C_p^2} \|\vartheta\|^2 - \|\nabla \varphi\|_{0,\beta}^2. \end{aligned} \quad (5.15)$$

Here we have used the inequality $(a - b)^2 \geq \frac{1}{2} a^2 - b^2$. Thus,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta_h\|^2 + \frac{c_4}{4C_p^2} \omega \|\theta_h\|^2 &\leq C \omega^{-1} \varepsilon^{-1} \|\vartheta_t\|^2 + \varepsilon \omega \|\theta_h\|^2 + C \omega \left(\|\vartheta\|^2 + \|\nabla \varphi\|_{0,\beta}^2 \right) \\ &\quad + C \Lambda(t) \left(\|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta} \right). \end{aligned}$$

Taking $\varepsilon = \frac{c_4}{8C_p^2}$, we find that

$$\frac{d}{dt} \|\theta_h\|^2 + \omega \|\theta_h\|^2 \leq C \omega^{-1} \|\vartheta_t\|^2 + C \omega \left(\|\vartheta\|^2 + \|\nabla \varphi\|_{0,\beta}^2 \right) + C \Lambda(t) \left(\|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta} \right). \quad (5.16)$$

Observing from (4.14) that $\omega(t) \leq 1$. Following from (4.14) and (4.58),

$$\omega^{-1}(t) \leq \left(1 + \|\nabla \rho\|_{0,\beta} + \|\nabla \rho_h\|_{0,\beta} \right)^{\beta\gamma} \leq (2^\beta \Lambda(t))^\gamma \leq 2^a \Lambda(t) \leq C \mathcal{F}(t). \quad (5.17)$$

Thus,

$$\frac{d}{dt} \|\theta_h\|^2 + 2^{-a} \Lambda(t)^{-1} \|\theta_h\|^2 \leq C \mathcal{F}(t) \left(\|\vartheta_t\|^2 + \|\vartheta\|^2 + \|\nabla \varphi\|_{0,\beta}^2 + \|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta} \right). \quad (5.18)$$

Applying Gronwall's inequality and using the fact that $\theta_h(0) = 0$, we obtain

$$\|\theta_h\|^2 \leq \int_0^t e^{-2^{-a} \int_s^t \Lambda(\tau)^{-1} d\tau} \mathcal{F}(s) \left(\|\partial_t\|^2 + \|\vartheta\|^2 + \|\nabla \varphi\|_{0,\beta}^2 + \|\nabla \varphi\|_{0,\beta} + \|\nabla \vartheta\|_{0,\beta} \right) ds. \quad (5.19)$$

Consequently,

$$\|\theta_h\|^2 \leq Ch^{k-1} \int_0^t \mathcal{F}(s) e^{-2^{-a} \int_s^t \Lambda(\tau)^{-1} d\tau} \left[\|\tilde{\rho}_t\|_k^2 + \|\tilde{\rho}\|_k^2 + \|\phi\|_{k,\beta}^2 + \|\phi\|_{k,\beta} + \|\tilde{\rho}\|_{k,\beta} \right] ds. \quad (5.20)$$

The inequality (5.7) follows by the triangle inequality and (5.20).

Applying Lemma 2.6 for (5.16), we obtain

$$\limsup_{t \rightarrow \infty} \|\theta_h\|^2 \leq C \limsup_{t \rightarrow \infty} \left[\omega^{-2} \|\partial_t\|^2 + \omega^{-1} \Lambda(t) \left(\|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta} \right) + \|\vartheta\|^2 + \|\nabla \varphi\|_{0,\beta}^2 \right].$$

Hence,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|\theta_h\|^2 &\leq Ch^{k-1} \limsup_{t \rightarrow \infty} \left[\Lambda(t)^2 \left(\|\tilde{\rho}_t\|_k^2 + \|\tilde{\rho}\|_{k,\beta} + \|\phi\|_{k,\beta} \right) + \|\tilde{\rho}\|_k^2 + \|\phi\|_{k,\beta}^2 \right] \\ &\leq Ch^{k-1} \left(\limsup_{t \rightarrow \infty} \Lambda(t) \right)^2 \limsup_{t \rightarrow \infty} \mathcal{H}(t) \\ &\leq Ch^{k-1} \mathcal{K}^2 \limsup_{t \rightarrow \infty} \mathcal{H}(t), \end{aligned}$$

which shows (5.9). The proof is complete. \square

The L^2 -error estimate and the inverse estimate enable us to have the L^∞ -error estimate as the following

COROLLARY 5.2. *Let $1 \leq k \leq r+1$, $\tilde{\rho}, \tilde{\rho}_h$ be solutions to (3.3) and (3.6) respectively. Assume that $\tilde{\rho} \in L^\infty(\mathbb{R}_+, W^{k,\infty}(\Omega))$, $\tilde{\rho}_t \in L^2(\mathbb{R}_+, H^k(\Omega))$. Then, there exists a constant positive constant C independent of h such that*

$$\|(\tilde{\rho} - \tilde{\rho}_h)(t)\|_{0,\infty}^2 \leq Ch^{2k} \|\tilde{\rho}_t\|_{k,\infty}^2 + Ch^{k-1-d} \int_0^t e^{-2^{-a} \int_s^t \Lambda(\tau)^{-1} d\tau} \mathcal{F}(s) \mathcal{H}(s) ds, \quad (5.21)$$

where $\mathcal{F}(t)$, $\mathcal{H}(t)$ are defined in (5.1) and (5.8) respectively.

Proof. Recall that in the quasi-uniform of \mathcal{T}_h we have the inverse estimate (see in [5, 33])

$$\|\theta_h\|_{0,\infty} \leq Ch^{-\frac{d}{2}} \|\theta_h\|.$$

This and triangle inequality imply that

$$\|\chi\|_{0,\infty}^2 \leq 2 \|\vartheta\|_{0,\infty}^2 + 2 \|\theta_h\|_{0,\infty}^2 \leq Ch^{2k} \|\tilde{\rho}\|_{k,\infty}^2 + Ch^{-d} \|\theta_h\|^2.$$

The inequality (5.21) follows directly from (5.20). \square

Now we give an error estimate for the gradient vector.

THEOREM 5.3. *Let $1 \leq k \leq r+1$. Assume that $\tilde{\rho} \in L^\infty(\mathbb{R}_+, H^k(\Omega))$, $\tilde{\rho}_t \in L^2(\mathbb{R}_+, H^k(\Omega))$. Let $\tilde{\rho}, \tilde{\rho}_h$ be solutions to (3.3) and (3.6) respectively. There exists a positive constants C independent of h such that for any $t \geq t_0 > 0$,*

$$\|\nabla(\rho - \rho_h)(t)\|_{0,\beta}^2 \leq Ch^{\frac{k-1}{2}} \mathcal{F}^2(t) \left(\left(\mathcal{G}(t) \int_0^t e^{-2^{-a} \int_s^t \Lambda(\tau)^{-1} d\tau} \mathcal{F}(s) \mathcal{H}(s) ds \right)^{\frac{1}{2}} + \mathcal{H}(t) \right), \quad (5.22)$$

where $\mathcal{F}(t), \mathcal{G}(t), \mathcal{H}(t)$ are defined in (4.15), (5.4), (5.8) respectively.

Furthermore, if $\int_0^\infty \Lambda^{-1}(t) dt = \infty$ then

$$\limsup_{t \rightarrow \infty} \|\nabla(\rho - \rho_h)(t)\|_{0,\beta}^2 \leq Ch^{\frac{k-1}{2}} \mathcal{K}^3 \mathcal{L} \left(\limsup_{t \rightarrow \infty} \mathcal{H}(t) + \left(\limsup_{t \rightarrow \infty} \mathcal{H}(t) \right)^{\frac{1}{2}} \right). \quad (5.23)$$

Proof. We rewrite equation (5.11) as the following

$$\begin{aligned} & (K(|\nabla \rho|)\nabla \rho - K(|\nabla \rho_h|)\nabla \rho_h, \nabla \rho - \nabla \rho_h) \\ &= (\rho_t - \rho_{h,t}, \theta_h) + (K(|\nabla \rho|)\nabla \rho - K(|\nabla \rho_h|)\nabla \rho_h, \nabla \vartheta + \nabla \varphi). \end{aligned} \quad (5.24)$$

According to (2.11),

$$(K(|\nabla \rho|)\nabla \rho - K(|\nabla \rho_h|)\nabla \rho_h, \nabla \rho - \nabla \rho_h) \geq c_4 \omega \|\nabla(\rho - \rho_h)\|_{0,\beta}^2. \quad (5.25)$$

Using Hölder's inequality and (5.13), we find that

$$\begin{aligned} & (\rho_t - \rho_{h,t}, \theta_h) + (K(|\nabla \rho|)\nabla \rho - K(|\nabla \rho_h|)\nabla \rho_h, \nabla \vartheta + \nabla \varphi) \\ & \leq C(|\rho_t| + |\rho_{h,t}|, |\theta_h|) + C(|\nabla \rho|^{\beta-1} + |\nabla \rho_h|^{\beta-1}, |\nabla \vartheta| + |\nabla \varphi|) \\ & \leq C(\|\rho_t\| + \|\rho_{h,t}\|) \|\theta_h\| + C\Lambda(t)(\|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta}). \end{aligned} \quad (5.26)$$

Combining (5.24) – (5.26), and (5.17) yields

$$\begin{aligned} \|\nabla(\rho - \rho_h)\|_{0,\beta}^2 & \leq C\omega^{-1}(\|\rho_t\| + \|\rho_{h,t}\|) \|\theta_h\| + C\omega^{-1}\Lambda(t)(\|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta}) \\ & \leq C\Lambda(t)^2 \left[(\|\rho_t\| + \|\rho_{h,t}\|) \|\theta_h\| + \|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta} \right] \\ & \leq C\mathcal{F}^2(t) \left[(\|\rho_t\| + \|\rho_{h,t}\|) \|\theta_h\| + \|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta} \right]. \end{aligned} \quad (5.27)$$

Due to (5.5), (5.20) and the fact that $\|\nabla \vartheta\|_{0,\beta} \leq Ch^{k-1} \|\tilde{\rho}\|_{k,\beta}$, the left hand side of (5.27) is bounded by

$$\begin{aligned} & C\mathcal{F}^2(t) \left\{ h^{k-1} \mathcal{G}(t) \int_0^t e^{-2^{-a} \int_s^t \Lambda(\tau)^{-1} d\tau} \mathcal{F}(s) \mathcal{H}(s) ds \right\}^{\frac{1}{2}} \\ & + Ch^{k-1} \mathcal{F}^2(t) (\|\tilde{\rho}\|_{k,\beta} + \|\phi\|_{k,\beta}). \end{aligned} \quad (5.28)$$

The inequality (5.22) follows from (5.27) and (5.28).

Take limit superior both sides of (5.27), we find that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \|\nabla(\rho - \rho_h)\|_{0,\beta}^2 \\ & \leq C \limsup_{t \rightarrow \infty} \Lambda(t)^2 \left[\limsup_{t \rightarrow \infty} (\|\rho_t\| + \|\rho_{h,t}\|) \limsup_{t \rightarrow \infty} \|\theta_h\| + \limsup_{t \rightarrow \infty} (\|\nabla \vartheta\|_{0,\beta} + \|\nabla \varphi\|_{0,\beta}) \right] \\ & \leq Ch^{\frac{k-1}{2}} \mathcal{K}^2 \left\{ \mathcal{K} \mathcal{L} \left(\limsup_{t \rightarrow \infty} \mathcal{H}(t) \right)^{\frac{1}{2}} + \limsup_{t \rightarrow \infty} (\|\tilde{\rho}\|_{k,\beta} + \|\phi\|_{k,\beta}) \right\}. \end{aligned}$$

Therefore, we obtain (5.23). \square

5.2. Error analysis for fully discrete Galerkin method. In analyzing this method, proceed in a similar fashion as for the semidiscrete method. We derive an error estimate for the fully discrete time Galerkin approximation of the differential equation. First, we give some uniform stability results that are crucial in getting the convergence results.

LEMMA 5.4. *Let $\tilde{\rho}_h^n$ solve the fully discrete Galerkin finite element approximation (3.7) for each time step $n \geq 1$. Then, there exists a positive constant C independent of $t, n, \Delta t$ satisfying*

$$\|\tilde{\rho}_h^n\| \leq C \max \left\{ \|\tilde{\rho}^0\|, 1 + \|\tilde{f}^n\|^{\frac{1}{p-1}} + \|\phi^n\|_1^{\frac{2}{p}} \right\}. \quad (5.29)$$

For all $i = 1 \dots n$,

$$\begin{aligned} \sum_{j=i}^n \Delta t \|\nabla \tilde{\rho}_h^j\|_{0,\beta}^\beta &\leq C \max \left\{ \|\tilde{\rho}^0\|^2, 1 + \|\tilde{f}^{i-1}\|^{\frac{2}{p-1}} + \|\phi^{i-1}\|_1^{\frac{4}{p}} \right\} \\ &\quad + C \sum_{j=i}^m \Delta t \left(1 + \|\tilde{f}^j\|^\lambda + \|\phi^j\|_1^2 \right). \end{aligned}$$

Proof. Choosing $w_h = 2\Delta t \tilde{\rho}_h^n$ in (3.7) and using identity

$$2(\tilde{\rho}_h^n - \tilde{\rho}_h^{n-1}, \tilde{\rho}_h^n) = \|\tilde{\rho}_h^n\|^2 - \|\tilde{\rho}_h^{n-1}\|^2 + \|\tilde{\rho}_h^n - \tilde{\rho}_h^{n-1}\|^2,$$

we obtain

$$\begin{aligned} \|\tilde{\rho}_h^n\|^2 - \|\tilde{\rho}_h^{n-1}\|^2 + \|\tilde{\rho}_h^n - \tilde{\rho}_h^{n-1}\|^2 + 2\Delta t \left\| K^{\frac{1}{2}}(|\nabla \rho_h^n|) \nabla \rho_h^n \right\|^2 \\ = 2\Delta t(\tilde{f}^n, \tilde{\rho}_h^n) + 2\Delta t(K(|\nabla \rho_h^n|) \nabla \rho_h^n, \nabla \pi \phi^n). \end{aligned}$$

Using Young's inequality and (4.7) then

$$2\Delta t(K(|\nabla \rho_h^n|) \nabla \rho_h^n, \nabla \pi \phi^n) \leq \frac{\Delta t}{2} \left\| K^{\frac{1}{2}}(|\nabla \rho_h^n|) \nabla \rho_h^n \right\|^2 + C\Delta t \|\nabla \pi \phi^n\|^2.$$

According to the inequality (4.8),

$$2\Delta t(\tilde{f}^n, \tilde{\rho}_h^n) \leq C\Delta t \|\tilde{f}^n\| + C\Delta t \|\tilde{f}^n\|^\lambda + \frac{\Delta t}{2} \left\| K^{\frac{1}{2}}(|\nabla \rho_h^n|) \nabla \rho_h^n \right\|^2 + C\Delta t \|\nabla \pi \phi^n\|^2.$$

Hence,

$$\|\tilde{\rho}_h^n\|^2 - \|\tilde{\rho}_h^{n-1}\|^2 + \|\tilde{\rho}_h^n - \tilde{\rho}_h^{n-1}\|^2 + \Delta t \left\| K^{\frac{1}{2}}(|\nabla \rho_h^n|) \nabla \rho_h^n \right\|^2 \leq C\Delta t \left(\|\tilde{f}^n\| + \|\tilde{f}^n\|^\lambda + \|\nabla \pi \phi^n\|^2 \right).$$

According to (4.10),

$$c_3 2^{-a} \left(2^{1-\beta} \|\nabla \tilde{\rho}_h^n\|_{0,\beta}^\beta - \|\nabla \pi \phi^n\|_{0,\beta}^\beta - 1 \right) \leq \left\| K^{\frac{1}{2}}(|\nabla \rho_h^n|) \nabla \rho_h^n \right\|^2. \quad (5.30)$$

From the two above inequalities, we obtain

$$\begin{aligned} \|\tilde{\rho}_h^n\|^2 - \|\tilde{\rho}_h^{n-1}\|^2 + \|\tilde{\rho}_h^n - \tilde{\rho}_h^{n-1}\|^2 + \frac{c_3}{2} \Delta t \|\nabla \tilde{\rho}_h^n\|_{0,\beta}^\beta \\ \leq C\Delta t \left(\|\tilde{f}^n\| + \|\tilde{f}^n\|^\lambda + \|\nabla \pi \phi^n\|^2 + \|\nabla \pi \phi^n\|_{0,\beta}^\beta + 1 \right) \\ \leq C\Delta t \left(1 + \|\tilde{f}^n\|^\lambda + \|\nabla \pi \phi^n\|^2 \right) \\ \leq C\Delta t \left(1 + \|\tilde{f}^n\|^\lambda + \|\phi^n\|_1^2 \right). \end{aligned} \quad (5.31)$$

Applying Poincaré inequality $\|\tilde{\rho}_h^n\| \leq C_p \|\nabla \tilde{\rho}_h^n\|_{0,\beta}$, shows that

$$\|\tilde{\rho}_h^n\|^2 - \|\tilde{\rho}_h^{n-1}\|^2 + \|\tilde{\rho}_h^n - \tilde{\rho}_h^{n-1}\|^2 + \frac{c_3 \Delta t}{2C_p} \|\tilde{\rho}_h^n\|^\beta \leq C \Delta t \left(1 + \|\tilde{f}^n\|^\lambda + \|\phi^n\|_1^2\right).$$

Applying the discrete Gronwall's version in Lemma 2.8, we find that

$$\|\tilde{\rho}_h^n\|^2 \leq C \max \left\{ \|\tilde{\rho}_h^0\|^2, (1 + \|\tilde{f}^n\|^\lambda + \|\phi^n\|_1^2)^{\frac{2}{\beta}} \right\},$$

which implies (5.29).

Now summing up (5.31) with n from i to m and dropping some nonnegative terms, we find that

$$\begin{aligned} \frac{c_3}{2} \sum_{j=i}^m \Delta t \|\nabla \tilde{\rho}_h^j\|_{0,\beta}^\beta &\leq \|\tilde{\rho}_h^{i-1}\|^2 + C \Delta t \sum_{j=i}^m \left(1 + \|\tilde{f}^j\|^\lambda + \|\phi^j\|_1^2\right) \\ &\leq C \max \left\{ \|\tilde{\rho}_h^0\|^2, (1 + \|\tilde{f}^{i-1}\|^\lambda + \|\phi^{i-1}\|_1^2)^{\frac{2}{\beta}} \right\} + C \sum_{j=i}^m \Delta t \left(1 + \|\tilde{f}^j\|^\lambda + \|\phi^j\|_1^2\right). \end{aligned}$$

The proof is complete. \square

As in the semidiscrete case, we use $\chi = \tilde{\rho} - \tilde{\rho}_h$, $\vartheta = \tilde{\rho} - \tilde{\rho}_h$, $\theta_h = \tilde{\rho}_h - \pi \tilde{\rho}$ and $\chi^n, \vartheta^n, \theta_h^n$ be evaluating $\chi, \vartheta, \theta_h$ at the discrete time levels. We also define

$$\partial \tilde{\rho}^n = \frac{\tilde{\rho}^n - \tilde{\rho}^{n-1}}{\Delta t}.$$

THEOREM 5.5. *Let $1 \leq k \leq r+1$, $\tilde{\rho}$ solve problem (3.3) and $\tilde{\rho}_h^n$ solve the fully discrete finite element approximation (3.7) for each time step n , $n \geq 1$. Suppose that $\tilde{\rho}_{tt} \in L^2(\mathbb{R}_+, L^2(\Omega))$, $\tilde{\rho} \in L^\infty(\mathbb{R}_+, H^k(\Omega))$. Then, there exists a positive constant $C(\rho)$ independent of h and Δt such that if the Δt is sufficiently small then*

$$\|\tilde{\rho}^n - \tilde{\rho}_h^n\|^2 \leq C(h^{k-1} + \Delta t). \quad (5.32)$$

Proof. Evaluating equation (3.3) at $t = t_n$ gives

$$(\tilde{\rho}_t^n, w) + (K(|\nabla \rho^n|) \nabla \rho^n, \nabla w) = (\tilde{f}^n, w), \quad \forall w \in W. \quad (5.33)$$

Subtracting (3.7) from (5.33), we obtain

$$(\partial \rho_h^n - \partial \pi \tilde{\rho}^n, w_h) + (K(|\nabla \rho_h^n|) \nabla \rho_h^n - K(|\nabla \rho^n|) \nabla \rho^n, \nabla w_h) = (\pi \tilde{\rho}_t^n - \partial \pi \tilde{\rho}^n, w_h). \quad (5.34)$$

We rewrite (5.34) as the form

$$(\partial \theta_h^n, w_h) + (K(|\nabla \rho_h^n|) \nabla \rho_h^n - K(|\nabla \rho^n|) \nabla \rho^n, \nabla w_h) = (\pi \tilde{\rho}_t^n - \partial \pi \tilde{\rho}^n, w_h). \quad (5.35)$$

Selecting $w_h = \theta_h^n$ in (5.35) gives

$$\begin{aligned} (\partial \theta_h^n, \theta_h^n) + (K(|\nabla \rho^n|) \nabla \rho^n - K(|\nabla \rho_h^n|) \nabla \rho_h^n, \nabla \rho^n - \nabla \rho_h^n) \\ = (K(|\nabla \rho^n|) \nabla \rho^n - K(|\nabla \rho_h^n|) \nabla \rho_h^n, \nabla \vartheta^n + \nabla \phi^n) + (\pi \tilde{\rho}_t^n - \partial \pi \tilde{\rho}^n, \theta_h^n). \end{aligned} \quad (5.36)$$

We will evaluate (5.36) term by term.

For the first term, we use the identity

$$(\partial\theta_h^n, \theta_h^n) = \left(\partial\theta_h^n, \frac{\theta_h^n + \theta_h^{n-1}}{2} + \frac{\Delta t}{2} \partial\theta_h^n \right) = \frac{1}{2\Delta t} \left(\|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2 \right) + \frac{\Delta t}{2} \|\partial\theta_h^n\|^2. \quad (5.37)$$

For the second term, the monotonicity of $K(\cdot)$ in (2.11) and (5.15) yield

$$\begin{aligned} (K(|\nabla\rho^n|)\nabla\rho^n - K(|\nabla\rho_h^n|)\nabla\rho_h^n, \nabla\rho^n - \nabla\rho_h^n) &\geq c_4\omega^n \|\nabla(\rho^n - \rho_h^n)\|_{0,\beta}^2 \\ &\geq \frac{c_4}{4C_p^2}\omega^n \|\theta_h^n\|^2 - \frac{c_4}{2C_p^2}\omega^n \|\vartheta^n\|^2 - \omega^n \|\nabla\varphi^n\|_{0,\beta}^2 \\ &\geq \frac{c_4}{4C_p^2}\omega^n \|\theta_h^n\|^2 - \frac{c_4}{2C_p^2}\|\vartheta^n\|^2 - \|\nabla\varphi^n\|_{0,\beta}^2. \end{aligned} \quad (5.38)$$

where

$$\omega^n = \omega(t_n) = \left(1 + \max \left\{ \|\nabla\rho_h^n\|_{0,\beta}, \|\nabla\rho^n\|_{0,\beta} \right\} \right)^{-a}.$$

For third term, using (5.13), we find that

$$(K(|\nabla\rho^n|)\nabla\rho^n - K(|\nabla\rho_h^n|)\nabla\rho_h^n, \nabla\vartheta^n + \nabla\varphi^n) \leq C\Lambda(t_n) \left(\|\nabla\vartheta^n\|_{0,\beta} + \|\nabla\varphi^n\|_{0,\beta} \right). \quad (5.39)$$

For the last term, it follows from using L^2 -projection and Taylor expand that

$$\begin{aligned} (\pi\rho_t^n - \partial\pi\rho^n, \theta_h^n) &= (\rho_t^n - \partial\rho^n, \theta_h^n) = \left(\frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \rho_{tt}(\tau)(\tau - t_{n-1}) d\tau, \theta_h^n \right) \\ &\leq \frac{1}{\Delta t} \left\| \int_{t_{n-1}}^{t_n} \rho_{tt}(\tau)(\tau - t_{n-1}) d\tau \right\| \|\theta_h^n\| \\ &\leq \frac{C}{\Delta t} \left(\int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau \right)^{\frac{1}{2}} \left(\int_{t_{n-1}}^{t_n} (\tau - t_{n-1})^2 d\tau \right)^{\frac{1}{2}} \|\theta_h^n\| \\ &\leq C\varepsilon^{-1} \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau + \varepsilon \|\theta_h^n\|^2. \end{aligned} \quad (5.40)$$

Combining (5.37), (5.38), (5.39) and (5.40), we obtain

$$\begin{aligned} \frac{1}{2\Delta t} \left(\|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2 \right) + \frac{c_4\omega^n}{4C_p^2} \|\theta_h^n\|^2 &\leq C\Lambda(t_n) \left(\|\nabla\vartheta^n\|_{0,\beta} + \|\nabla\varphi^n\|_{0,\beta} \right) \\ &\quad + C\varepsilon^{-1} \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau + \varepsilon \|\theta_h^n\|^2 + C \|\vartheta^n\|^2 + C \|\nabla\varphi^n\|_{0,\beta}^2. \end{aligned}$$

Choosing $\varepsilon = \frac{c_4\omega^n}{8C_p^2}$ in previous inequality, we find that

$$\begin{aligned} \frac{1}{2\Delta t} \left(\|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2 \right) + \frac{c_4\omega^n}{8C_p^2} \|\theta_h^n\|^2 &\leq C\Lambda(t_n) \left(\|\nabla\vartheta^n\|_{0,\beta} + \|\nabla\varphi^n\|_{0,\beta} \right) \\ &\quad + C(\omega^n)^{-1} \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau + C \|\vartheta^n\|^2 + \|\nabla\varphi^n\|_{0,\beta}^2 \\ &\leq C\Lambda(t_n) \left(\|\nabla\vartheta^n\|_{0,\beta} + \|\nabla\varphi^n\|_{0,\beta} + \|\vartheta^n\|^2 + \|\nabla\varphi^n\|_{0,\beta}^2 + \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau \right). \end{aligned}$$

According to discrete Gronwall's inequality in Lemma 2.8,

$$\begin{aligned}
\|\theta_h^n\|^2 &\leq C(\omega^n)^{-1} \Lambda(t_n) \left(\|\nabla \vartheta^n\|_{0,\beta} + \|\nabla \varphi^n\|_{0,\beta} + \|\vartheta^n\|^2 + \|\nabla \varphi^n\|_{0,\beta}^2 + \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau \right) \\
&\leq C\Lambda(t_n)^2 \left(\|\nabla \vartheta^n\|_{0,\beta} + \|\nabla \varphi^n\|_{0,\beta} + \|\vartheta^n\|^2 + \|\nabla \varphi^n\|_{0,\beta}^2 + \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau \right) \\
&\leq C\mathcal{F}^2(t_n) \left(\|\nabla \vartheta^n\|_{0,\beta} + \|\nabla \varphi^n\|_{0,\beta} + \|\vartheta^n\|^2 + \|\nabla \varphi^n\|_{0,\beta}^2 + \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau \right).
\end{aligned} \tag{5.41}$$

The inequality (5.32) follows from combining (5.41) and the inequality

$$\|\bar{\rho}^n - \bar{\rho}_h^n\|^2 \leq 2\|\theta_h^n\|^2 + 2\|\vartheta^n\|^2. \tag{5.42}$$

The proof is complete. \square

THEOREM 5.6. *Under assumption of Theorem 5.5. There exists a positive constant $C(\rho, \phi)$ independent of h and Δt such that if the Δt is sufficiently small then*

$$\|\nabla \rho_h^n - \nabla \rho^n\|_{0,\beta}^2 \leq C(h^{k-1} + \Delta t). \tag{5.43}$$

Proof. We rewrite (5.34) with $w_h = \theta_h$ as

$$\begin{aligned}
&(K(|\nabla \rho^n|)\nabla \rho^n - K(|\nabla \rho_h^n|)\nabla \rho_h^n, \nabla \rho^n - \nabla \rho_h^n) \\
&= (K(|\nabla \rho^n|)\nabla \rho^n - K(|\nabla \rho_h^n|)\nabla \rho_h^n, \nabla \vartheta^n + \nabla \varphi^n) + (\rho_t^n - \partial \rho^n, \theta_h^n).
\end{aligned}$$

Due to (5.38), (5.39) and (5.40), we have

$$\begin{aligned}
c_4 \omega^n \|\nabla \rho_h^n - \nabla \rho^n\|_{0,\beta}^2 &\leq C\Lambda(t_n) \left(\|\nabla \vartheta^n\|_{0,\beta} + \|\nabla \varphi^n\|_{0,\beta} \right) \\
&\quad + C\varepsilon^{-1} \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau + \varepsilon \|\theta_h^n\|^2.
\end{aligned}$$

Using (5.38), we find that

$$\varepsilon \|\theta_h^n\|^2 \leq 4\varepsilon C_p^2 \|\nabla \rho_h^n - \nabla \rho^n\|_{0,\beta}^2 + C\varepsilon(\omega^n)^{-1} \left(\|\vartheta^n\|^2 + \|\nabla \varphi^n\|_{0,\beta}^2 \right).$$

Hence,

$$\begin{aligned}
c_4 \omega^n \|\nabla \rho_h^n - \nabla \rho^n\|_{0,\beta}^2 &\leq C\Lambda(t_n) \left(\|\nabla \vartheta^n\|_{0,\beta} + \|\nabla \varphi^n\|_{0,\beta} \right) + C\varepsilon^{-1} \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau \\
&\quad + 4\varepsilon C_p^2 \|\nabla \rho_h^n - \nabla \rho^n\|_{0,\beta}^2 + C\varepsilon(\omega^n)^{-1} \left(\|\vartheta^n\|^2 + \|\nabla \varphi^n\|_{0,\beta}^2 \right).
\end{aligned} \tag{5.44}$$

Selecting $\varepsilon = \frac{c_4 \omega^n}{8C_p^2}$ in previous inequality gives

$$\begin{aligned}
\|\nabla \rho_h^n - \nabla \rho^n\|_{0,\beta}^2 &\leq C\Lambda(t_n)(\omega^n)^{-1} \left(\|\nabla \vartheta^n\|_{0,\beta} + \|\nabla \varphi^n\|_{0,\beta} \right) \\
&\quad + C(\omega^n)^{-2} \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau + C(\omega^n)^{-1} \left(\|\vartheta^n\|^2 + \|\nabla \varphi^n\|_{0,\beta}^2 \right).
\end{aligned}$$

Then using (5.17), we have $(\omega^n)^{-1} \leq \Lambda(t_n)$. Note that $\Lambda(t_n) > 1$, thus

$$\begin{aligned}
\|\nabla \rho_h^n - \nabla \rho^n\|_{0,\beta}^2 &\leq C\Lambda(t_n)^2 \left(\|\nabla \vartheta^n\|_{0,\beta} + \|\nabla \varphi^n\|_{0,\beta} + \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau + \|\vartheta^n\|^2 + \|\nabla \varphi^n\|_{0,\beta}^2 \right) \\
&\leq C\mathcal{F}^2(t_n) \left(\|\nabla \vartheta^n\|_{0,\beta} + \|\nabla \varphi^n\|_{0,\beta} + \Delta t \int_{t_{n-1}}^{t_n} \|\rho_{tt}(\tau)\|^2 d\tau + \|\vartheta^n\|^2 + \|\nabla \varphi^n\|_{0,\beta}^2 \right).
\end{aligned}$$

This proves (5.43). The proof is complete. \square

6. Numerical results. In this section, we give simple numerical experiments using Galerkin finite element method in the two dimensional region to illustrate the convergent theory. We test the convergence of our method with the Forchheimer two-term law $g(s) = 1 + s$. Equation (1.13) $sg(s) = \xi$, $s \geq 0$ gives $s = \frac{-1 + \sqrt{1+4\xi}}{2}$ and hence

$$K(\xi) = \frac{1}{g(s(\xi))} = \frac{2}{1 + \sqrt{1+4\xi}}.$$

Example 1. The analytical solution is as follows

$$\rho(x, t) = e^{-2t} x_1(1 - x_1)x_2(1 - x_2), \quad \forall (x, t) \in [0, 1]^2 \times [0, 1].$$

The forcing term f is determined accordingly to the analytical solution by equation $p_t - \nabla \cdot (K(|\nabla \rho|)\nabla \rho) = f$. Explicitly,

$$\begin{aligned} f(x, t) = & -2e^{-2t} x_1(1 - x_1)x_2(1 - x_2) + \frac{4e^{-2t} [x_2(1 - x_2) + x_1(1 - x_1)]}{1 + \sqrt{1 + 4e^{-2t} w(x)}} \\ & + \frac{2e^{-4t} x_2(1 - x_2)(1 - 2x_1) [2x_1(1 - x_1)^2(1 - 2x_2)^2 - 2x_1^2(1 - x_1)(1 - 2x_2)^2 - 4x_2^2(1 - x_2)^2(1 - 2x_1)]}{w(x)\sqrt{1 + 4e^{-2t} w(x)} \left(1 + \sqrt{1 + 4e^{-2t} w(x)}\right)^2} \\ & + \frac{2e^{-4t} x_1(1 - x_1)(1 - 2x_2) [2x_2(1 - x_2)^2(1 - 2x_1)^2 - 2x_2^2(1 - x_2)(1 - 2x_1)^2 - 4x_1^2(1 - x_1)^2(1 - 2x_2)]}{w(x)\sqrt{1 + 4e^{-2t} w(x)} \left(1 + \sqrt{1 + 4e^{-2t} w(x)}\right)^2}, \end{aligned}$$

where $w(x) = \sqrt{(x_2(1 - x_2)(1 - 2x_1))^2 + (x_1(1 - x_1)(1 - 2x_2))^2}$.

The initial data $\rho^0(x) = x_1(1 - x_1)x_2(1 - x_2)$ and the boundary data $\psi(x, t) = 0$.

We use the Lagrange element of order $r = 1$ on the unit square in two dimensions. Our problem is solved at each time level starting at $t = 0$ until the given final time T . At time T , we measured the error in L^2 -norm for density and L^β -norm for the gradient density. In this example $\beta = 2 - a = 2 - \frac{\deg(g)}{\deg(g)+1} = \frac{3}{2}$. The numerical results are listed in Table I.

At time $T = 1$				
N	$\ \rho - \rho_h\ $	Rates	$\ \nabla(\rho - \rho_h)\ _{0,\beta}$	Rates
4	1.668E-02	-	7.081E-02	-
8	1.049E-02	0.669	4.654E-02	0.605
16	6.004E-03	0.805	2.741E-02	0.764
32	3.272E-03	0.876	1.530E-02	0.841
64	1.723E-03	0.926	8.277E-03	0.887
128	8.889E-04	0.954	4.411E-03	0.908
256	4.531E-04	0.972	2.336E-03	0.917
At time $T = 10$				
N	$\ \rho - \rho_h\ $	Rates	$\ \nabla(\rho - \rho_h)\ _{0,\beta}$	Rates
4	1.851E-02	-	7.818E-02	-
8	1.194E-02	0.633	5.247E-02	0.575
16	6.892E-03	0.792	3.114E-02	0.753
32	3.782E-03	0.866	1.752E-02	0.830
64	2.001E-03	0.918	9.537E-03	0.877
128	1.035E-03	0.951	5.104E-03	0.902
256	5.280E-04	0.971	2.713E-03	0.912

Table I. Convergence study for generalized Forchheimer flows using Galerkin FEM with zero boundary data in 2D.

Example 2. The analytical solution is $\rho(x, t) = e^{1-t}(x_1^2 + x_2^2)$ for all $(x, t) \in [0, 1]^2 \times [0, 1]$. The forcing term f , initial condition and boundary condition are determined accordingly to the analytical solution as follows

$$f(x, t) = -e^{1-t}z(x) + \frac{16e^{2-2t}z(x)}{\sqrt{z(x)}\sqrt{1+8e^{1-t}\sqrt{z(x)}}\left(1+\sqrt{1+8e^{1-t}\sqrt{z(x)}}\right)^2} - \frac{8e^{1-t}}{1+\sqrt{1+8e^{1-t}\sqrt{z(x)}}},$$

$$\rho^0(x) = e \cdot z(x), \quad \psi(x, t) = e^{1-t} \begin{cases} x_2^2 & \text{on } x_1 = 0, \\ 1 + x_2^2 & \text{on } x_1 = 1, \\ 1 + x_1^2 & \text{on } x_2 = 1, \\ x_1^2 & \text{on } x_2 = 0, \end{cases}$$

where $z(x) = x_1^2 + x_2^2$. The numerical results are listed in Table II

At time $T = 1$				
N	$\ \rho - \rho_h\ $	Rates	$\ \nabla(\rho - \rho_h)\ _{0,\beta}$	Rates
4	$7.785E-02$	-	$3.767E-01$	-
8	$5.700E-02$	0.450	$2.972E-01$	0.342
16	$2.870E-02$	0.990	$1.918E-01$	0.631
32	$2.296E-02$	0.322	$1.038E-01$	0.887
64	$1.434E-02$	0.679	$6.029E-02$	0.783
128	$5.459E-03$	1.393	$3.641E-02$	0.728
256	$2.401E-03$	1.185	$2.128E-02$	0.775
At time $T = 10$				
N	$\ \rho - \rho_h\ $	Rates	$\ \nabla(\rho - \rho_h)\ _{0,\beta}$	Rates
4	$9.335E-06$	-	$5.502E-05$	-
8	$6.829E-06$	0.451	$4.050E-05$	0.442
16	$4.983E-06$	0.455	$2.345E-05$	0.788
32	$3.014E-06$	0.726	$1.381E-05$	0.764
64	$1.351E-06$	1.158	$8.231E-06$	0.747
128	$6.447E-07$	1.067	$4.759E-06$	0.790
256	$3.935E-07$	0.712	$2.666E-06$	0.836

Table II. Convergence study for generalized Forchheimer flows using Galerkin FEM with nonzero Dirichlet boundary data in 2D.

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